Problem 3. Minors and duals.

Let M be a matroid on E and let $A \subseteq E$. I will use these 3 formulas:

① $r_{M\backslash T}(X) = r_M(X)$

② $r_{M/T}(X) = r_M(X \cup T) - r_M(T)$

③ $r_{M^*} = |X| + r_M(E - X) - r_M$

To show the following:

(a) $(M/A)^* = M^*\backslash A$

(proof.) Let $X \subseteq E - A$, then
$r_{(M/A)^*}(X) = |X| + r_{(M/A)}(E - A - X) - r_{M/A}(E - A)$ ③
$= |X| + r_M((E - A - X) \cup A) - r_M(A) - (r_M((E - A) \cup A) - r_M(A))$ ②
$= |X| + r_M(E - X) - r_M(E)$
Also

$r_{M^*\backslash A}(X) = r_{M^*}(X) =$ ①
$|X| + r_M(E - X) - r_M(E)$ ③
So $r_{(M/A)^*}(X) = r_{M^*\backslash A}(X)$, thus $(M/A)^* = M^*\backslash A$ □

(b) $cl_{M/A}(X) = cl_M(X \cup A) - A$ for all $X \subseteq E - A$.

(proof.) $\subseteq$ Let $y \notin A$, $y \in cl_{M/A}(X)$ so $r_{M/A}(X) = r_{M/A}(X \cup y)$ⓐ
$r_{M/A}(X \cup y) = r_M(X \cup y \cup A) - r_M(A)$ⓑ
Also, $r_{M/A}(X) = r_M(X \cup A) - r_M(A)$ⓒ

By ⓐ ⓑ ⓒ $r_M(X \cup y \cup A) = r_M(X \cup A) = r_M(X \cup A \cup y)$, so $y \in cl_M(X \cup A)$ and since $y \notin A$, $y \in cl_M(X \cup A) - A$. Thus $cl_{M/A}(X) \subseteq cl_M(X \cup A) - A$

$\supseteq$ I want to show $r_{M\backslash A}(X) = r_{M\backslash A}$
$r_M(X) - r_M(A) = r_M(X \cup y) - r_M(A)$

Let $y \in cl_M(X \cup A) - A$ so $y \notin A$
$r_M(X \cup A \cup y) = r_M(X \cup A)$
WLOG Suppose $X \cap A = \emptyset$ so $(X \cup A) - A = X$
$r_{M/A}(X \cup A) = r_M(X \cup A - A) - r_M(A)$ ②
$= r_M(X) - r_M(A)$ⓘ

$r_{M/A}(X \cup A \cup y) = r_M(X \cup A \cup y - A) - r_M(A)$ ②

$= r_M(X \cup y) - r_M(A)$ⓘⓘ

By ⓘ and ⓘⓘ $r_M(X \cup A \cup y) - r_M(A) = r_M(X \cup A) - r_M(A)$

This is what I needed to show, so $y \in cl_M(X \cup A) - A \Rightarrow y \in cl_{M/A}(X)$
Thus $cl_M(X \cup A) - A \subseteq cl_{M/A}(X)$

So $cl_{M/A}(X) = cl_M(X \cup A) - A$ □

(c) M/A has no loops if and only if A is a flat of M.

(proof.) M/A has no loops $\Rightarrow A$ is a flat in M.

Suppose A is not a flat in M, then $r_M(A \cup y) = r_M(A)$ so $r_M(A \cup y) - r_M(A) = 0$

$r_{M/A}(y) = r_M(A \cup y) - r_M(A) = 0$②
$r_M(A \cup y) = r_M(A)$ So y is a loop in M/A.$\Rightarrow\Leftarrow$

M/A has no loops $\Leftarrow A$ is not a flat in M.

Suppose x is a loop of M/A.

Then $r_{M/A}(x) = 0$
$r_{M/A}(x) = r_M(x \cup A) - r_M(A) = 0$ ②

$r_M(x \cup A) = r_M(A)$

So A is not a flat if M/A has loops. Thus, if A is a flat, then M/A has no loops. □