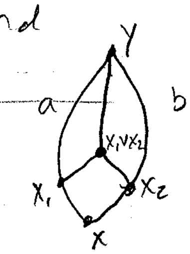
2 Prove that a finite lattice $\mathcal{L}$ is semimodular iff whenever x and y both cover $x \wedge y$ then $x \vee y$ covers both x and y.

"$\Longrightarrow$" Suppose $\mathcal{L}$ is semimodular, and x and y both cover $x \wedge y$. Then $r(x) = r(y) = r(x \wedge y) + 1$. By semimodularity $r(x) + r(y) \geq r(x \vee y) + r(x \wedge y)$,
so $r(x \wedge y) + 1 + r(x \wedge y) + 1 \geq r(x \vee y) + r(x \wedge y) \Longrightarrow r(x \vee y) - r(x \wedge y) \leq 2$.
If $x \vee y$ didn't cover x and y, then $x \vee y$ would have to be more than 2 floors higher than $x \wedge y$. $\#$

"$\Longleftarrow$" Suppose the cover relation holds. We will show that $\mathcal{L}$ is graded and that $r(x) + r(y) \geq r(x \vee y) + r(x \wedge y)$.

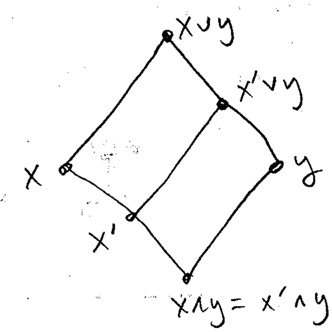
$\underline{\mathcal{L} \text{ is graded}}$: Suppose $\mathcal{L}$ is not graded. Therefore there are elements x and y in $\mathcal{L}$ between whom exist chains of different length. Since $\mathcal{L}$ is finite, let x and y be minimal with this condition — i.e. the interval $[x, y]$ is of minimal length. Let a and b be two chains of differing lengths from x to y. Let $x_1 \in a$, $x_2 \in b$ be covers of x. In particular, by minimality assumption, $x_1 \neq x_2$ and so $x_1 \wedge x_2 = x$. Then by hypothesis $x_1 \vee x_2$ covers both x_1 and x_2.

Now observe that $[x_1, y]$ and $[x_2, y]$ are both strictly shorter chains than $[x, y]$, hence by our minimality assumption these two intervals must be graded. In particular, the chain along a from x_1 to y — call it a' — must be the same length as the chain from x_1 through $x_1 \vee x_2$ up to y. Since $x_1 \vee x_2$ covers both x_1 and x_2, then this must be the same length as the chain from x_2 through $x_1 \vee x_2$ up to y. And finally this will be the same length as the chain b' of x_2 to y along b. So a' and b' have the same length, and hence a and b also have the same length. Contradiction!



<u>$\mathcal{L}$ satisfies the semimodularity inequality:</u>

Suppose it doesn't, i.e. suppose that $\exists\, x, y \in \mathcal{L}$ such that $r(x) + r(y) < r(x \vee y) + r(x \wedge y)$. Pick x and y such that the length of the chain from $x \wedge y$ to $x \vee y$ is minimal, and that $r(x) + r(y)$ is minimal.



If x and y cover $x \wedge y$ then the result swiftly follows by our cover relation assumption. So assume x and y don't cover $x \wedge y$. So without loss of generality let's take x' between x and $x \wedge y$. By minimality of x and y, we must have $r(x') + r(y) \geq r(x' \wedge y) + r(x' \vee y)$. Let's add these two inequalities:

$$(r(x) + r(y)) + (r(x' \wedge y) + r(x' \vee y)) < (r(x \vee y) + r(x \wedge y)) + (r(x') + r(y)).$$

Since $x \wedge y = x' \wedge y$ we can simplify this inequality to

$$r(x) + r(x' \vee y) < r(x') + r(x \vee y).$$

We can see by the picture that $x \wedge (x' \vee y) \geq x'$ and $x \vee (x' \vee y) = x \vee y$. To make things simpler, let $a = x$ and $b = (x' \vee y)$. Then using the previous inequality we get

$$\begin{array}{ccccccc} r(x) & + & r(x' \vee y) & < & r(x') & + & r(x \vee y) \\ \| & & \| & & \wedge\!| & & \| \\ r(a) & + & r(b) & < & r(a \wedge b) & + & r(a \vee b), \end{array}$$

But $a \vee b$ and $a \wedge b$ are closer together than $x \vee y$ and $x \wedge y$, so we've contradicted the minimality of the chain from $x \wedge y$ to $x \vee y$. This completes the proof!