Anna Brown
Math 890.02
March 16, 2007

Matroids Homework 4

1. **The partition lattice.** A *partition* of $[n]$ is a collection $\pi = \{S_1, ..., S_k\}$ of pairwise disjoint subsets of $[n]$ (called the *blocks* of π) whose union is $[n]$. Consider the set of partitions of $[n]$, with the following partial order: If π_1 and π_2 are partitions of $[n]$, say that $\pi_1 \leq \pi_2$ if every block of π_2 is a union of blocks of π_1.

   (a) Prove that this defines a poset of Π_n.

   Proof. Let π_i, π_j, π_k be partitions of $[n]$. Note for all i, any block π_i is the union of itself. Thus $\pi_i \leq \pi_i$ for all $\pi_i \in \Pi_n$.

   Suppose $\pi_i \leq \pi_j$ and $\pi_j \leq \pi_i$. Since $\pi_i \leq \pi_j$, every block in π_j is a union of blocks of π_i, which is a union of blocks of π_j (since $\pi_j \leq \pi_i$). So π_i and π_j have the same blocks, and thus $\pi_i = \pi_j$.

   Suppose $\pi_i \leq \pi_j$ and $\pi_j \leq \pi_k$. Since $\pi_j \leq \pi_k$, every block in π_k is a union of blocks of π_j. But every block in π_j is a union of blocks of π_i (since $\pi_i \leq \pi_j$). So every block in π_k is a union of blocks of π_i, and thus $\pi_i \leq \pi_k$. □

   (b) Prove that Π_n is a lattice.

   Proof. Let $\pi_i, \pi_j \in \Pi_n$, where $\pi_i = \{A_1, A_2, ..., A_l\}$ and $\pi_j = \{B_1, B_2, ..., B_m\}$, where A_i and B_j are blocks of π_i and π_j respectively.

   Claim: $\pi_m = \{A_i \cap B_j | 1 \leq i \leq l, 1 \leq j \leq m, A_i \cap B_j \neq \emptyset\}$ is $\pi_i \wedge \pi_j$.

   Proof. Note that for $i \neq i'$, $A_i \cap A_{i'} = \emptyset$ and for $j \neq j'$, $B_j \cap B_{j'} = \emptyset$. Also, $\cup_{i,j}(A_i \cap B_j) = \cup_i A_i \cap \cup_j B_j = [n] \cap [n] = [n]$. Thus $\pi_m \in \Pi_n$.

   Note also that since each $A_i \cap B_j \subseteq$ and $A_i \cap B_j \subseteq B_j$, $\pi_m \leq \pi_i$ and $\pi_m \leq \pi_j$. So π_m is a lower bound of π_i and π_j. Now choose some other lower bound $\pi'_m \in \Pi_n$ such that $\pi'_m \leq \pi_i$ and $\pi'_m \leq \pi_j$. Since $\pi'_m \in \Pi_n$, it is of the form $\pi'_m = \{C_1, C_2, ..., C_k\}$, where each C_n is a block of π'_m. Then for each C_n, $1 \leq n \leq k$, there exists some i such that $C_n \subseteq A_i$ and some j such that $C_n \subseteq B_j$. Thus for that choice of i and j, $C_n \subseteq A_i \cap B_j$, which means $\pi'_m \leq \pi_m$. So π_m is the greatest lower bound, and thus $\pi_m = \pi_i \wedge \pi_j$. □

   Now consider the set of all upper bounds for π_i and π_j in Π_n. So for all upper bounds π_u, $\pi_i \leq \pi_u$ and $\pi_j \leq \pi_u$. By the above claim and proof, we know that the meet π_M of all the upper bounds exists and is in Π_n. Note also that for any two upper bounds π_u and π'_u, $\pi_i \leq \pi_u$ and $\pi_i \leq \pi'_u$. So $\pi_i \leq \pi_u \wedge \pi'_u$. Similarly, $\pi_j \leq \pi_u \wedge \pi'_u$. By the same reasoning, if we consider a third upper bound, π''_u, then $\pi_i \leq (\pi_u \wedge \pi'_u) \wedge \pi''_u$ and $\pi_j \leq (\pi_u \wedge \pi'_u) \wedge \pi''_u$. If we continue in this manner, we see that π_i and π_j are both less than or equal to π_M, the meet of all the upper bounds of Π_n. So π_M is an upper bound of π_i and π_j. And since π_M itself is less than or equal to all the upper bounds of π_i and π_j, it is the least upper bound. Thus $\pi_M = \pi_i \vee \pi_j$.

   Since Π_n is a poset (by 1a, above) and has a greatest lower bound and least upper bound, it is a lattice. □

(c) Prove that Π_n is graded, and describe its rank function.

Proof. Note that Π_n is graded if and only if there exists a rank function r such that π_i is minimal in Π_n means $r(\pi_i) = 0$ and that if π_j covers π_i, $r(\pi_j) = \pi_i + 1$. Define $r(\pi_i) = n - \{\text{number of blocks in } \pi_i\}$. Let π_i be minimal in Π_n. Since we have a lattice, π_i is unique, and by the way we've constructed Π_n, π_i contains n partitions of single elements. So $r(\pi_i) = n - n = 0$.

Now suppose that π_j covers π_i. Then $\pi_i < \pi_j$, which means there is no $\pi_k \in \Pi_n$ such that $\pi_i < \pi_k < \pi_j$ and every block of π_j is a union of blocks of π_i. Note that if we union more than 2 blocks of π_i, there would be a $\pi_k \in \Pi_n$ that would have a block that contains the union of some (but not all) of these blocks of π_i. But this would mean that $\pi_i < \pi_k < \pi_j$, which isn't possible since π_j covers π_i. So the blocks of π_j are exactly the same as the blocks of π_i, except that exactly two blocks of π_i have been unioned together to create a single block in π_j. So we lost exactly one block in going up from π_i to π_j. Thus $r(\pi_j) = n - (\{\text{number of blocks in } \pi_i\} - 1) = r(\pi_i) + 1$.

Since for our rank function r, π_i is minimal in $\Pi_n \Rightarrow r(\pi_i) = 0$ and π_j covers $\pi_i \Rightarrow r(\pi_j) = \pi_i + 1$, Π_n is graded. □

(d) Prove that Π_n is semimodular.

Proof. Let $\pi_i, \pi_j \in \Pi_n$ both cover $\pi_i \wedge \pi_j$. Then $r(\pi_i) = r(\pi_i \wedge \pi_j) + 1$ and $r(\pi_j) = r(\pi_i \wedge \pi_j) + 1$. So $r(\pi_i) = r(\pi_j)$. Let $\pi_i \wedge \pi_j = \{S_1, S_2, ..., S_k\}$, where the S_i $(1 \leq i \leq k)$ are blocks. Now without loss of generality, suppose $\pi_i = \{S_1 \cup S_2, S_3, ..., S_k\}$.

Case 1: Without loss of generality, suppose $\pi_j = \{S_1 \cup S_3, S_2, S_4, ..., S_k\}$. Then $\pi_i \vee \pi_j = \{S_1 \cup S_2 \cup S_3, S_4, ..., S_k\}$. Since there is one less block in $\pi_i \vee \pi_j$ than there is in π_i and π_j, $r(\pi_i \vee \pi_j) = r(\pi_i) + 1 = r(\pi_j) + 1$.

Case 2: Without loss of generality, suppose $\pi_j = \{S_1, S_2, S_3 \cup S_4, ..., S_k\}$. Then $\pi_i \vee \pi_j = \{S_1 \cup S_2, S_3 \cup S_4, S_5, ..., S_k\}$, and once again we have one less block in $\pi_i \vee \pi_j$ than there in π_i and π_j. So $r(\pi_i \vee \pi_j) = r(\pi_i) + 1 = r(\pi_j) + 1$.

Thus in both cases, $r(\pi_i \vee \pi_j) = r(\pi_i) + 1 = r(\pi_j) + 1$, which means $\pi_i \vee \pi_j$ covers both π_i and π_j. Since $\pi_i, \pi_j \in \Pi_n$ both covering $\pi_i \wedge \pi_j \Rightarrow \pi_i \vee \pi_j$ covers both π_i and π_j, by Problem 2 on this assignment, Π_n is semimodular. □

(e) Prove that Π_n is atomic.

Proof. Note that the atoms of Π_n are the lattice elements with rank 1. Since we defined the rank of an element in Π_n as $r(\pi_i) = n - \{\text{number of blocks in } \pi_i\}$, any element with rank 1 has $n - 1$ blocks. So every atom of Π_n has one block of two numbers (where these numbers are between 1 and n) and $n - 2$ blocks of 1 number. In fact, since each element of Π_n (and thus each atom of Π_n) consists of disjoint partitions, our atoms are uniquely determined by its 2-block.

Then for all $\pi_i \in \Pi_n$, if π_i contains a block that contains some set of 2 numbers, say ij, where $i \neq j$ and $1 \leq i \leq n$ and $1 \leq j \leq n$, then one of the atoms we joined to get to π_i was the atom containing ij as its 2-block. By our cover relation, every block of π_i is the union of blocks below it. So we can break every block in π_i that contains more than 1 element into the union of 2-blocks. (If a block of π_i contains an odd number of elements, we can break it into the union of 2 blocks and one 1 block.) And since there exists an atom containing each of these 2-blocks, we can join these atoms to build π_i. Thus every $\pi_i \in \Pi_n$ is a join of a unique set of atoms. □

(f) Prove that Π_n is the lattice of flats of $M(K_n)$, the graphical matroid of the complete graph K_n.

Claim: Each $\pi_i \in \Pi_n$ corresponds to a flat of $M(K_n)$.

Proof. Choose some $\pi_i \in \Pi_n$ and randomly number the vertices of K_n. Let Φ be defined as the function that maps a block of π_i to the complete graph on the subsets of $[n]$ contained in each block. For example, if $S_i = \{1, 3, 5, 6\}$ is a block of π_i, then $\Phi(S_i)$ would be the complete graph on the vertices 1, 3, 5, 6. For any block S_j of 2 elements, $\Phi(S_j)$ gives a single edge between those two vertices, and for any block S_k consisting of a single element, $\Phi(S_k)$ is just a single vertex in K_n. So for any $\pi_i \in \Pi_n$, $\Phi(\pi_i)$ (where we apply Φ to every block of π_i) is a subgraph of K_5.

Now let's add an edge to $\Phi(\pi_i)$. Since the blocks of π_i are disjoint, Φ of each block of π_i is not connected to the graphs of Φ of the other blocks of π_i. So if we add an edge to $\Phi(\pi_i)$, we won't complete a circuit, because the map of each block of π_i is already a complete graph and thus contains all its circuits. So adding an edge to $\Phi(\pi_i)$ will increase its rank, which means $cl(\Phi(\pi_i)) = \Phi(\pi_i)$, and thus $\Phi(\pi_i)$ is a flat of $M(K_n)$. (Note that if $\pi_i = \hat{1}$, then $\Phi(\hat{1}) = K_n$. There are no edges to add, and so trivially, $cl(\Phi(\hat{1})) = \Phi(\hat{1})$.)

Now consider the other direction. Let every flat of $M(K_5)$ correspond to a collection of complete subgraphs $\{G_1, G_2, ..., G_n\}$ of K_5. Then $\Phi^{-1}(\{G_1, G_2, ..., G_n\}) = \{S_1, S_2, ..., S_n\}$, where each S_i is the collection of vertices of the subgraphs of each G_i. So $\{S_1, S_2, ..., S_n\}$ is a partition of the n vertices of K_5.

Each $\pi_i \in \Pi_n$ corresponds to a flat of $M(K_n)$, and each flat of $M(K_n)$ corresponds to a partition of Π_n. Thus Φ is an isomorphism between Π_n and $M(K_n)$, which means Π_n is the lattice of flats of $M(K_n)$. $\square$