(4) "$\Leftarrow$" Let's prove $M_1 = M_2^*$ if the partition condition holds. Let $B \in \mathcal{B}_1$ be a base $\Rightarrow cl_1(B) = E$. Recall that

$I \in \mathcal{I}_{e} \Leftrightarrow x \notin cl_*(I-x)\ \forall x \in I$. For $e \in E-B$,

The partition $(\{e\}, B, E-B-e)$ yields $e \notin cl_2(E-B-e) \Rightarrow$

$E-B \in \mathcal{I}_2$. (*) [~~Suppose $A \supset E-B$ is a base of M_2. Applying the same argument $E-A \in \mathcal{I}_1$. Now notice that $E-A \in B$ and $E-B \subset A \Rightarrow E-B = A \Rightarrow E-B$ is a base of $M_2 \Rightarrow$~~]

B is a base of M_2^*. The same argument can be applied for $B \in \mathcal{B}_2^* \Rightarrow B \in (\mathcal{B}_1^*)^* \Rightarrow B \in \mathcal{B}_1 \Rightarrow M_1 = M_2^*$

(*) Now let's prove $cl_2(E-B) = E$. Let $x \in cl_2(E-B)$ and $x \in B$.
If $x \notin cl_2(E-B) \Rightarrow x \in cl_1(B-x)$ which is impossible since B is a base $\Rightarrow x \in cl_2(E-B) \Rightarrow cl_2(E-B) = E$.
Since $E-B \in \mathcal{I}_{e_2}$ and $cl_2(E-B) = E \Rightarrow E-B \in \mathcal{B}_2$ (If not then for $A \in \mathcal{B}_2$ $A \supset E-B$, $a \in A-(E-B)$ we have $a \notin cl_2(E-B)$) $\Rightarrow B$ base for M_2^*

"$\Rightarrow$" Suppose $M_1 = M_2^*$. Let $(\{e\}, X, Y)$ be a disjoint partition of $E \Rightarrow$ Let $B_1 \in \mathcal{B}_{1,X}$ and $B_2 \in \mathcal{B}_{2,Y}$.

If $e \notin cl_1(X) \Rightarrow B_1 \cup e \in \mathcal{B}_{1, X\cup\{e\}}$. Suppose $e \notin cl_2(Y)$

$B_2 \cup e \in \mathcal{B}_{2, Y\cup\{e\}}$. Extend both by B_3 and B_4.

so $B_1 \cup e \cup B_3 \in \mathcal{B}_1$ and $B_2 \cup e \cup B_4 \in \mathcal{B}_2$.

$\Rightarrow (B_2 \cup e \cup B_4)^c \in \mathcal{B}_2^* \Rightarrow |E| - |B_2| - |e| - |B_4| = |B_1| + |e| + |B_3|$. This implies $|B_1| + |B_2| + |B_3| + |B_4| + 2|e| = |E|$

and since $|B_1| + |B_3| = |X|$ and $|B_2| + |B_4| = |Y|$.

$\Rightarrow |X| + |Y| + 2 = |E| \Rightarrow\Leftarrow$. In the same manner

if $e \in cl_1(X) \cap cl_2(Y) \Rightarrow |X| + |Y| = |E| \Rightarrow\Leftarrow$.

So e belongs to exactly one of $cl_1(X)$ or $cl_2(Y)$