**3** Suppose that $r : 2^E \to \mathbb{N}$ satisfies the standard axioms R1, R2, R3 for a rank function. Then R1′ for $r$ follows from R1 with $X = \emptyset$. The lower bound $r(A \cup \{a\} - r(A) \geq 0$ in R2′ follows from R2 with $Y = A \cup \{a\}$, $X = A$; the upper bound $r(A \cup \{a\}) - r(A) \leq 1$ is a consequence of R3 with $X = A$, $Y = \{a\}$, bounding the $r(Y)$ and $r(\emptyset)$ that then appear by R1. Finally, R3′ comes from R3 in the situation $X = A \cup \{a\}$, $Y = A \cup \{b\}$, where the bound from R3 is $r(A \cup \{a\} \cup \{b\}) \leq r(A)$, which is equality by R2.

Conversely suppose that $r : 2^E \to \mathbb{N}$ satisfies our local axioms R1′ ,R2′ ,R3′. We'll show R1, R2, R3 using telescoping sum techniques. To show R1, the lower bound $0 \leq r(X)$ is a trivial consequence of the range we've defined $r$ with. For the upper bound, label the elements of $X$ as $x_1, \ldots, x_n$, and write $X_k = \{x_1, \ldots, x_k\}$ for $0 \leq k \leq n$; then by R1′ and R2′, we have

$$r(X) = r(X_n) = r(X_0) + \sum_{k=1}^{n} r(X_k) - r(X_{k-1}) \leq 0 + \sum_{k=1}^{n} 1 = n,$$

which is R1.

For R2 we do similarly. Given $X$ and $Y \supseteq X$, write $Y = X \cap \{y_1, \ldots, y_n\}$, and put $Y_k = X \cap \{y_1, \ldots, y_k\}$. Then by R2′,

$$r(Y) - r(X) = r(Y_k) - r(Y_0) = \sum_{k=1}^{n} r(Y_k) - r(Y_{k-1}) \geq \sum_{k=1}^{n} 0 = 0,$$

which is R2.

Finally, for R3, we first note that R3′ can be restated, assuming R2′, to assert that

$$r(A \cup \{a\}) + r(A \cup \{b\}) - r(A) - r(A \cup \{a, b\}) \geq 0.$$

For if either $r(A \cup \{a\}) - r(A)$ or $r(A \cup \{b\}) - r(A)$ is 1, then this condition is vacuous in light of R2′; but if $r(A \cup \{a\}) - r(A) = r(A \cup \{b\}) - r(A) = 0$, this condition is equivalent to R3′. Now given sets $X$ and $Y$, put $X \setminus Y = \{x_1, \ldots, x_n\}$, $Y \setminus X = \{y_1, \ldots, y_m\}$, and $Z_{k,l} := (X \cap Y) \cup \{x_1, \ldots, x_k\} \cup$

$\{y_1, \ldots, y_l\}$. Then, using R3′,

$$\begin{aligned}
&r(X) + r(Y) - r(X \cap Y) - r(X \cup Y) \\
&= r(Z_{n,0}) + r(Z_{0,m}) - r(Z_{0,0}) - r(Z_{n,m}) \\
&= \sum_{k=1}^{n} \big(r(Z_{k,0}) - r(Z_{k-1,0})\big) - \big(r(Z_{k,m}) - r(Z_{k-1,m})\big) \\
&= \sum_{k=1}^{n} \sum_{l=1}^{m} r(Z_{k,l}) - r(Z_{k,l-1}) - r(Z_{k-1,l}) + r(Z_{k-1,l-1}) \\
&\geq \sum_{k,l} 0 = 0,
\end{aligned}$$

which is R3.