matroid theory

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homework five

Note. You are encouraged to work together on the homework, but you must state who you worked with. You *must* write your solutions independently and in your own words.

Nota. Los estudiantes en Bogotá pueden escoger cuatro de los cinco problemas. El quinto vale puntos adicionales.

1. Representability of $U_{2,n}$. Over which fields is the uniform matroid $U_{2,n}$ representable?
2. Representability over a field of characteristic zero has infinitely many forbidden minors. Let $p > 2$ be a prime number and let $\mathbb{F}_p$ be the field of p elements. Let L_p be the vector matroid of the following configuration of $2(p+1)$ vectors in $\mathbb{F}_p^{p+1}$:

$$V_p = \{(1,0,\ldots,0,0),(0,1,\ldots,0,0),\ldots,(0,0,\ldots,1,0),(0,0,\ldots,0,1),$$

$$(0,1,\ldots,1,1),(1,0,\ldots,1,1),\ldots,(1,1,\ldots,0,1),(1,1,\ldots,1,0)\}$$

   (a) Prove that L_p is not representable over any field of characteristic 0.

   (b) Prove that L_p is not a minor of L_q for $p < q$.

3. (optional) The non-Desargues matroid is not algebraic. Prove it.
4. Computing Möbius functions. Let P and Q be posets. Let $P \times Q$ be the poset whose elements are the pairs (p, q) such that $p \in P$ and $q \in Q$, and whose order relation is given by:

$$(p_1, q_1) \leq_{P\times Q} (p_2, q_2) \quad \text{if and only if} \quad p_1 \leq_P p_2 \text{ and } q_1 \leq_Q q_2.$$

   (a) Prove that the Möbius function of $P \times Q$ is given by

$$\mu_{P\times Q}(p, q) = \mu_P(p)\mu_Q(q).$$

   (b) Find the Möbius function of the Boolean lattice $2^{[n]}$.

   (c) Find the Möbius function of the lattice D_n of divisors of n.

   (d) Find the Möbius function of the partition lattice Π_n.

5. Counting proper colorings and acyclic orientations of graphs. Let G be a graph and M be its graphical matroid. Let c be the number of connected components of G.

   (a) Let q be a positive integer. Prove that $q^c\chi_M(q)$ is the number of ways of coloring the vertices of G with q given colors in such a way that neighboring vertices have different colors.

   (b) Prove that $|\chi_M(-1)|$ is the number of ways of orienting each edge of G in such a way that no directed cycles are formed.