matroid theory

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homework four

Note. You are encouraged to work together on the homework, but you must state who you worked with. You *must* write your solutions independently and in your own words.

1. The partition lattice. A *partition* of $[n]$ is a collection $\pi = \{S_1, \ldots, S_k\}$ of pairwise disjoint subsets of $[n]$ (called the *blocks* of π) whose union is $[n]$. Consider the set of partitions of $[n]$, with the following partial order: If π_1 and π_2 are partitions of $[n]$, say that $\pi_1 \leq \pi_2$ if every block of π_2 is a union of blocks of π_1.

   (a) Prove that this defines a poset Π_n.

   (b) Prove that Π_n is a lattice.

   (c) Prove that Π_n is graded, and describe its rank function.

   (d) Prove that Π_n is semimodular.

   (e) Prove that Π_n is atomic.

   (f) Prove that Π_n is the lattice of flats of $M(K_n)$, the graphical matroid of the complete graph K_n.

   Note. Solving part (f) would immediately solve the other ones, since the lattice of flats of a matroid is geometric. However, the purpose of this exercise is to get your hands dirty and really get acquainted with the partition lattice, so I want you to solve (a)-(e) directly from the definitions.

2. Semimodular lattices. Prove that a finite lattice $\mathcal{L}$ is semimodular if and only if it satisfies the following condition:

   If $x, y \in \mathcal{L}$ are such that x and y both cover $x \wedge y$, then $x \vee y$ covers both x and y.

3. Minors and duals. Let M be a matroid on E and let $A \subseteq E$. Show the following:

   (a) $(M/A)^* = M^* \backslash A$

   (b) $\mathrm{cl}_{M/A}(X) = \mathrm{cl}_M(X \cup A) - A$ for all $X \subseteq E - A$.

   (c) M/A has no loops if and only if A is a flat of M.

4. Parallel elements in cotransversal matroids. Show that if e and f are parallel elements in a cotransversal matroid M, then $M \backslash e$ is also cotransversal.

5. The matroid of bases of minimum weight. Let $M = (E, \mathcal{B})$ be a matroid and let $w : E \to \mathbb{R}$ be a weight function on E. For each real number r, let $E_r = \{e \in E \mid w(e) \leq r\}$. Notice that there are only finitely many different sets E_r; let's call them $S_1, \ldots, S_k$.

   Let M_w be the matroid of bases of minimum weight of M. Find a description of M_w in terms of the matroid M, the sets S_i, direct sums, and minors.