matroid theory

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homework three

1. Duality in graphs. Let G be a connected plane graph and G^* its dual.

   (a) An *isthmus* is an edge whose removal disconnects the graph. Prove that e is a loop in G if and only if e^* is an isthmus in G^*.

   (b) A *bond* is a minimal set of edges whose removal disconnects the graph. Prove that C is a circuit of G if and only if C^* is a bond of G^*.

   (c) Let H be the graph obtained by removing edge e from G. Describe the graph H^* in terms of G^*.

2. Non-graphical duals of graphical matroids. Let K_5 be the complete graph on 5 vertices, and let $K_{3,3}$ be the complete bipartite graph with 3 left vertices and 3 right vertices. Prove that the matroids $M(K_5)^*$ and $M(K_{3,3})^*$ are not graphical.

   (If you've tried for a while and still don't know how to get started, see Oxley, Prop. 2.3.3.)

3. A "local" axiom system for rank functions. Prove that a function $r : 2^E \to \mathbb{N}$ is the rank function of a matroid on E if and only if r satisfies the following conditions:

(R1)$'$ $r(\emptyset) = 0$

(R2)$'$ If $A \subseteq E$ and $a \in E$ then $r(A \cup a) - r(A) = 0$ or 1.

(R3)$'$ If $A \subseteq E$ and $a, b \in E$ satisfy that $r(A \cup a) = r(A \cup b) = r(A)$, then $r(A \cup a \cup b) = r(A)$.

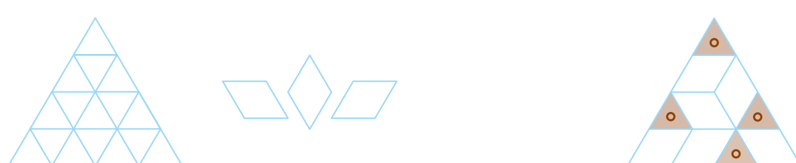
4. A closure characterization of dual matroids. Let M_1 and M_2 be matroids on a set E, and let cl_1 and cl_2 be their closure operators. Prove that $M_1 = M_2^*$ if and only if, for every partition $(\{e\}, X, Y)$ of E into three disjoint sets, the element e is in exactly one of $\mathrm{cl}_1(X)$ and $\mathrm{cl}_2(Y)$.

5. Cotransversal matroids are matroids. Let G be a graph with directed edges, and let X and Y be two sets of vertices with $|X| = |Y| = r$. A *routing* from X to Y is a collection of r directed paths such that the initial vertex of each path is in X, the final vertex of each path is in Y, and no two paths have a vertex in common. Note that a path could consist of a single vertex, if this vertex is in both X and Y.

   Let B_0 be a set of vertices of G. Let $L(G, B_0)$ be the collection of subsets X for which there exists a routing from X to B_0. Prove, directly from the basis axioms, that $L(G, B_0)$ is the collection of bases of a matroid.

6. Matroids from tilings.

   (a) You are given an equilateral triangular board of size n divided into little unit triangles, and tiles which are little $60° - 120°$ unit rhombi. Notice that it is impossible to tile the board with the given tiles, because the board contains $\binom{n+1}{2}$ triangles facing up and $\binom{n}{2}$ facing down, and each unit rhombus must cover one triangle of each kind.



   However, if you punch $\binom{n+1}{2} - \binom{n}{2} = n$ unit triangular holes, you may (or may not) be able to tile the resulting board. If it is possible to tile it, we will call the set of n holes *good*. The picture shows a good set of 4 holes, and a corresponding tiling.

   Prove that the good sets of holes are the bases of a matroid.

   (b) Is the same result true for any board consising of little unit equilateral triangles?