matroid theory

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homework two

Note. You are encouraged to work together on the homework, but you must state who you worked with. You *must* write your solutions independently and in your own words.

1. Parallelism in matroids.

   In the matroid on $\{a, b, c, d, e, f\}$ with bases $\{abc, abd, abe, acd, ace\}$, which we discussed on the first day of class, we said that the elements d and e seemed special because they were "parallel". In this exercise I want you to make this precise.

   (a) Come up with two good (and equivalent) definitions of parallelism in matroids; one in terms of independent sets, and one in terms of bases:
   Two elements d and e of a matroid $M = (E, \mathcal{I})$ are *parallel* if ____________________.
   Two elements d and e of a matroid $M = (E, \mathcal{B})$ are *parallel* if ____________________.

   (b) Prove that your two definitions are equivalent.

   (c) Your definitions should be such that parallelism is an equivalence relation. Prove that.

   You may want to take into account the following discussion:
   `http://circulo.uniandes.edu.co/preliminar/phpBB2/viewtopic.php?t=29`

2. Many different matroids.

   Let $1 \leq a_1 < a_2 < \cdots < a_r \leq n$ be positive integers. Let $\mathcal{B}_{a_1,\ldots,a_r}$ consists of the sets $\{b_1 < \cdots < b_r\}$ such that $b_i \leq a_i$.

   (a) Prove that $\mathcal{B}_{a_1,\ldots,a_r}$ is the collection of bases of a matroid on $[n]$.

   (b) Prove that these 2^n matroids are non-isomorphic.

   (c) Can you find a better lower bound on the number of non-isomorphic matroids on $[n]$?

3. Independent sets and spanning sets.

   A *spanning set* of a matroid on E is a subset of E which contains a basis. If an independent set I is contained in a spanning set S, prove that there exists a basis B such that $I \subseteq B \subseteq S$.

4. Bases of minimum weight.

   Let $M = (E, \mathcal{B})$ be a matroid and $w : E \to \mathbb{R}_{>0}$ be a weight function.

   (a) If $w(e) \neq w(f)$ for $e \neq f$, prove that M has a unique basis of minimum weight.

   (b) For any w, prove that the bases of minimum weight are the bases of a matroid M_w.

5. Matroids as simplicial complexes.

   An *abstract simplicial complex* $(E, \mathcal{I})$ (sometimes simply called a *simplicial complex*) consists of a finite set E and a family $\mathcal{I} \subseteq 2^E$ of subsets of E satisfying axioms (I1) and (I2) of a matroid. It is called *pure* simplicial complex if every maximal set in $\mathcal{I}$ has the same size. The *restriction* of $\mathcal{I}$ to a subset $A \subseteq E$ is the simplicial complex $(A, \mathcal{I}_A)$, where

   $$\mathcal{I}_A = \{I \in \mathcal{I} \mid I \subseteq A\}.$$

   Prove that a simplicial complex $(E, \mathcal{I})$ is a matroid if and only if its restriction to any subset of E is pure.

   (Note. If you know and/or are interested in topology, you should ask yourself: What can we say about the topology of this simplicial complex? This could be an interesting final project.

6. Symmetric exchange.

   Let A and B be bases of a matroid M, and $a \in A - B$. Prove that there exists $b \in B - A$ such that $A - a \cup b$ and $B - b \cup a$ are bases of M.