matroid theory

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homework one

Note. You are encouraged to work together on the homework, but you must state who you worked with. You *must* write your solutions independently and in your own words.

0. Register in the online discussion forum, following the instructions on the course website.
1. Linear matroids as matroids of subspaces.
   (a) Let A and B be two $r \times n$ matrices such that $\text{rowspace}(A) = \text{rowspace}(B)$. Prove that if we regard the columns of A and the columns of B as vector configurations, then these two configurations give isomorphic matroids.
   (b) By part (a), we can define the matroid of a subspace of a vector space with respect to a fixed system of coordinates. Prove that a matroid is linear if and only if it is isomorphic to the matroid of a subspace.
2. Linear matroids as matroids of hyperplane arrangements.
   A *hyperplane* in a vector space V is a subspace of V of codimension 1. Let $\mathcal{A}$ be an *arrangement of hyperplanes* in V. Say that k hyperplanes in $\mathcal{A}$ are *independent* if their intersection has codimension k. Let $\mathcal{I}(\mathcal{A})$ be the collection of independent subsets of $\mathcal{A}$.
   (a) Prove that $(\mathcal{A}, \mathcal{I}(\mathcal{A}))$ is a matroid.
   (b) Prove that a matroid is linear if and only if it is isomorphic to the matroid of a hyperplane arrangement.
3. Graphical matroids are linear.
   Let $G = (V, E)$ be a graph and $M(G)$ be its matroid. Let $\mathbb{F}$ be any field, and let $\mathbb{F}^V$ be the $\mathbb{F}$-vector space having standard basis vectors x_v indexed by the vertices v of G. To each edge $e = \{a, b\}$ of the graph, assign the vector $v_e = x_a - x_b$. Prove that the matroid of the configuration of vectors v_e is isomorphic to $M(G)$.
4. Graphical and transversal uniform matroids.
   Let $r \leq n$ be positive integers. The *uniform matroid* $U_{r,n}$ is the matroid whose ground set has n elements, and whose independent sets are all subsets of size less than or equal to r.
   (a) For which r, n is the uniform matroid $U_{r,n}$ graphical?
   (b) For which r, n is the uniform matroid $U_{r,n}$ transversal?
5. Transversal matroids are matroids.
   Let G be a bipartite graph with vertex bipartition (L, R). Say that a subset I of R *can be matched to* L if there exist $|I|$ edges whose left endpoints are distinct and whose right endpoints are the vertices in I. Let $\mathcal{I}$ be the collection of subsets of R which can be matched to L. Prove that $(R, \mathcal{I})$ is a matroid.
6. A matroid of paths.
   A *Dyck path* of length $2n$ is a path in the plane from $(0,0)$ to $(2n, 0)$, with steps $U = (1,1)$ and $D = (1,-1)$, that never passes below the x-axis. For example, $P = UUDUDUUDDD$ is a Dyck path of length 10. Each Dyck path defines an *up-step set*: the subset of $[2n]$ consisting of the integers i such that the i-th step of the path is U. The up-step set of P is $\{1, 2, 3, 6, 7\}$. Let $\mathcal{B}_n$ be the collection of up-step sets of the Dyck paths of length $2n$. Prove that $\mathcal{B}_n$ is the collection of bases of a matroid.