Combinatorial Hopf Algebras

A **combinatorial Hopf algebra** (CHA) $(\mathcal{H},\alpha)$ is a graded connected Hopf algebra $\mathcal{H}=\bigoplus_{n\geq 0}\mathcal{H}_n$ over $\mathbb{K}$ with $\dim(\mathcal{H}_n)$ finite for all n, together with a character $\alpha:\mathcal{H}\to\mathbb{K}$

(linear, multiplicative)

Exs:

- $\mathcal{H}$ = Hopf alg of graphs

  product: disjoint union

  coproduct: $\Delta(G)=\sum_{W\subseteq V} G|_W\otimes G|_{V-W}$

  character: $\alpha(G)=\begin{cases}1 & G=\circ\ \circ\ \circ\ \circ\ \circ\ \circ\\ 0 & \text{ow}\end{cases}$

- $\mathcal{H}$ = Hopf alg of posets

  prod: disjoint union

  coprod: $\Delta(P)=\sum_{\substack{I\subseteq P\\ \text{order ideal}}} I\otimes(P-I)$

  character: $\alpha(P)=1$ (all P)

- $\mathcal{H}$ = QSym

  character: $\zeta_Q(M_\alpha)=\begin{cases}1 & \alpha=n \text{ or } \emptyset\\ 0 & \text{ow}\end{cases}$

  (plug in $(1,0,0,\ldots)$)

  $\mathcal{H}$ = Sym

  $\zeta_S(m_\lambda)=\begin{cases}1 & \lambda=n,\ \emptyset\\ 0 & \text{ow}\end{cases}$

A **CHA morphism** $f:(H,\alpha)\to(K,\beta)$ is a graded Hopf alg. morphism $f:H\to K$ s.t. this commutes:

$$\begin{tikzcd} H \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & K \arrow[dl, "\beta"] \\ & \mathbb{K} & \end{tikzcd} \qquad \alpha(h)=\beta(f(h))$$

Theorem (Aguiar-Bergeron-Sottile)
For any CHA (H,α) there is a unique CHA morphism $\Psi:(H,\alpha)\to(\mathrm{QSym},\zeta_Q)$

I.e., (QSym,ζ_Q) is the terminal object in the category of CHAs

Pf

Let $\alpha:\mathcal{H}\to\mathbb{K}$ restrict to $\alpha_n:\mathcal{H}_n\to\mathbb{K}$, so $\alpha_n\in\mathcal{H}^*$.

There is a unique algebra map

$$\varphi:\mathrm{NSym}\longrightarrow\mathcal{H}^*$$
$$h_n\longmapsto\alpha_n$$

since NSym is free. This gives a dual coalgebra map

$$\Psi:\mathcal{H}\longrightarrow\mathrm{QSym}=\mathrm{NSym}^*$$

which, we claim, is the one we are after.

Check: it is an algebra map also.

Duality means

$$\langle \Psi(f), g\rangle = \langle f, \Psi^*(g)\rangle = \langle f, \Psi(g)\rangle$$

$\underbrace{f \in NSym}_{H^*}$, $g \in \mathcal{H}$, $f \in NSym$, $\underbrace{\Psi(g) \in QSym}_{\uparrow H}$

For $f = h_n$,

$$\langle \alpha_n, g\rangle = \langle h_n, \Psi(g)\rangle$$

$$\alpha_n(g) = h_n(\Psi(g)) = \zeta_Q(\Psi(g))$$

equal in deg n: send $\begin{cases} M_n \text{ to } 1 \\ M_\lambda \text{ to } 0 \quad \lambda \neq n \end{cases}$

So Ψ preserves the character. (And any morphism preserving the character must be dual to Ψ, conversely) ∎

Since

$$\alpha_c(g) = \langle h_c, \Psi(g)\rangle$$

where

$\alpha_c = \alpha_{c_1} \cdots \alpha_{c_k}(g)$ in $\mathcal{H}^*$ $\qquad h_c = h_{c_1} \cdots h_{c_k}$ in NSym

$$\boxed{\alpha_c = \sum_{(g)} \alpha_{c_1}(g_{(1)}) \cdots \alpha_{c_k}(g_{(k)})}$$

if we write $\Psi(g) = \sum_d \mu_d M_d$ then

$$\alpha_c(g) = \langle h_c, \textstyle\sum_d \mu_d M_d\rangle = \mu_c$$

so

$$\boxed{\Psi(g) = \sum_c \alpha_c(g) M_c} \quad \text{for } g \in \mathcal{H}$$

is the formula for the morphism

Restricting to cocomm:

Theorem
For any cocommutative CHA $(\mathcal{H}, \alpha)$ there is a unique CHA morphism

$$\Psi : (\mathcal{H}, \alpha) \longrightarrow (Sym, \zeta_S)$$

The proof is similar, and relies on

$$Sym^* = \mathbb{K}[h_1, h_2, \ldots]$$

(free commutative)

Here the formula is

$$\boxed{\Psi(g) = \sum_\lambda \alpha_\lambda(g) m_\lambda} \quad \text{for } g \in \mathcal{H}.$$

So we can use our knowledge of QSym and Sym, and their connections to repn theory of S_n, to study any combinatorial objects having a CHA structure

Ex

$\mathcal{H}$ = Hopf alg of graphs (cocomm.)

$$\Delta(G) = \sum_{W \subseteq V} G|_W \otimes G|_{V-W} \qquad G \cdot H = G\,H$$

$$\alpha(G) = \begin{cases} 1 & G = \bullet\ \bullet\ \bullet\ \bullet\ \bullet \\ 0 & \text{ow.} \end{cases}$$

Then

$$\Psi(G) = \sum_{\lambda \vdash n} \alpha_\lambda(G)\, m_\lambda$$

Here

$$\alpha_\lambda(G) = \sum_{V = W_1 \cup \cdots \cup W_k} \alpha_{\lambda_1}(W_1) \cdots \alpha_{\lambda_k}(W_k)$$

= # of partitions of V into sets W_i ("colors") with no $\underset{W_i\ \ W_i}{\bullet\!-\!\bullet}$ edges with $|W_i| = \lambda_i$

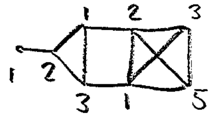
= # of proper colorings of the graph G using color i exactly λ_i times.

So

$$\boxed{\Psi(G) = \sum_{\substack{K: V \to \mathbb{N} \\ \text{proper}}} x_K} \quad \text{where } x_K := \prod_{v \in V} x_{K(v)}$$

G graph

$$\boxed{X_G = \sum_{\substack{K: V \to \mathbb{N} \\ \text{proper}}} x_K} \qquad x_K := \prod_{v \in V} x_{K(v)}$$



$$x_K = x_1^3 x_2^2 x_3^2 x_5$$

This is Stanley's chromatic symmetric function, which connects graph coloring to rep'n theory.

Ex:

- $G = K_n$

$$X_G = \sum_{\substack{K: [n] \to \mathbb{N} \\ \text{proper}}} x_K = \sum_{\substack{K: [n] \to \mathbb{N} \\ K(i) \neq K(j)}} x_{K(1)} \cdots x_{K(n)}$$

$$\boxed{X_{K_n} = e_n} \quad \text{elem. symm. fn}$$

- $G =$ 

$$X_G = 24 \sum x_i x_j x_k x_l + 6 \sum x_i^2 x_j x_k + \sum x_i^3 x_j$$

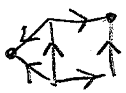
$$= 4e_4 + 5e_{31} - 2e_{22} + e_{211}$$

Prop (Stanley)

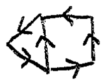
If $X_G = \sum_{\lambda \vdash n} c_\lambda e_\lambda$, then

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} c_\lambda = \overbrace{\begin{pmatrix} \#\text{ of acyclic orientations of } G \\ \text{with } k \text{ sinks} \end{pmatrix}}^{a_k(G)}$$

An acyclic orientation is an assignment of directions to the edges which induces no cycles.



yes
(2 sinks)



no

A sink of an a.o. is a vertex with no outgoing edges.

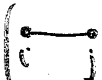
Ex $G =$ 

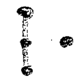
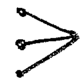
3 sinks | 2 sinks (3 of these) | 1 sink (3 of these) | 1 sink

$a_3(G)=1$ $\quad a_2(G)=3$ $\quad a_1(G)=4$ $\quad a(G)=8$

Q When is X_G e-positive? ($c_\lambda \geq 0$)

- Let P be a poset which is (3+1)-free (No induced subposet is )
- The incomparability graph G_P of P has

$\left(\right.$  $\left. \in G_P \right) \iff \begin{pmatrix} i \not< j \\ i \not> j \end{pmatrix}$ in P

$P =$  $\Rightarrow G =$ 

Conjecture (Stanley, 1995)

If P is (3+1)-free, then $X_{G(P)}$ is e-positive

Known: True if P is 3-free

Prop There is a morphism of Hopf algebras

$$\mathrm{QSym} \longrightarrow \mathbb{F}[x]$$

$$F(x_1, x_2, \dots) \longmapsto P_F(k) = F(\underbrace{1, \dots, 1}_{k}, 0, 0, \dots)$$

Ex:

$$F = e_n = \sum_{i_1 \neq \dots \neq i_n} x_{i_1} \cdots x_{i_n}$$

$$P_F(k) = \sum_{i_1 \neq \dots \neq i_n} \begin{Bmatrix} 1 & \text{if } i_1 \dots i_n \leq k \\ 0 & \text{otherwise} \end{Bmatrix} = \binom{k}{n}$$

$$P_F(x) = \frac{x(x-1)\cdots(x-n+1)}{n!}$$

Corollary

For any CHA $(\mathcal{H}, \alpha)$ there is a morphism

$$\mathcal{H} \to \mathbb{F}[x]$$

$$h \mapsto \chi_h(k) = \sum_{(h)} \alpha(h_{(1)}) \cdots \alpha(h_{(k)}) = \alpha^k(h)$$

mult. in $\mathcal{H}^*$ $\downarrow$

Pf This is just

$$\mathcal{H} \longrightarrow \mathrm{QSym} \longrightarrow \mathbb{F}[x] \qquad \blacksquare$$

Ex: G graph

$$\alpha(G) = \begin{cases} 1 & G = \cdot\ \cdot\ \cdot\ \cdot \\ 0 & \text{otherwise} \end{cases}$$

$$\chi_G(k) = \sum \begin{cases} 1 & \text{if all } G_i \text{ are } \cdot\ \cdot\ \cdot\ \cdot \\ 0 & \text{o/w} \end{cases}$$

$$\boxed{\chi_G(k) = \#\text{ of proper colorings of } G \text{ with } k \text{ colors}}$$

The chromatic polynomial!

$\Rightarrow$ indeed a polynomial

Prop

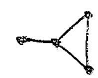
$$\boxed{\chi_G(q) = q\,\chi_{L_G}(q)}$$

Chromatic $\uparrow$ $\uparrow$ characteristic

Ex P poset

$$\alpha(P) = 1$$

$$\chi_P(k) = \sum 1$$



$$q(q-1)^2(q-2) = q^4 - 4q^3 + 5q^2 - 2q$$

Thm (Huh 2011)
Coeffs are unimodal: go up, then down

(1968 Conjecture!)

$$\boxed{\chi_P(k) = \#\text{ of } f : P \to [k] \text{ such that } i < j \Rightarrow f(i) \leq f(j)}$$

The weak order polynomial!

$\Rightarrow$ indeed a polynomial

$$\boxed{\begin{array}{l}\underline{\text{Prop}}\text{ (Combinatorial Reciprocity)}\\ \chi_h(-n) = \chi_{S(h)}(n)\\ \chi_h(-1) = \alpha(S(h))\end{array}}$$

$$\begin{tikzcd} \mathcal{H} \arrow[r] \arrow[d, "S"] & \mathbb{F}[x] \arrow[d, "S"', "\binom{S(f(x))}{=f(-x)}"] \\ \mathbb{K} \arrow[r, equal] & \mathbb{K} \end{tikzcd}$$

Ex:

① Graphs:

$$S(G) = \sum_{F \text{ flat}} (-1)^{|V| - rk(F)} a(G/F)\, G|_F$$

Apply α:

$\left(\text{Note: } \alpha(G|_F) \neq 0 \iff G|_F \text{ has no edges} \iff F = \emptyset\right)$

$$\alpha(S(G)) = (-1)^{|V|} a(G)$$

$$\boxed{\begin{array}{l}\underline{\text{Theorem}}\text{ (Stanley)}\\ \chi_G(-1) = (-1)^{|V|} a(G)\end{array}}$$

② Posets:

$\overline{\Omega}_P(k) = \#$ of $f: P \to [k]$ with $i < j \Rightarrow f(i) \leq f(j)$

$\Omega_P(k) = \#$ of $f: P \to [k]$ with $i < j \Rightarrow f(i) < f(j)$

$$\boxed{\begin{array}{l}\underline{\text{Theorem}}\\ \Omega_P(-n) = \\ (-1)^{|P|} \overline{\Omega}_P(n)\end{array}}$$