Matroids

Matroids are a combinatorial model of independence.

A <u>matroid</u> $M = (E, \mathcal{I})$ is a set E with a collection $\mathcal{I}$ of subsets of E, called "independent sets", such that

- $\emptyset \in \mathcal{I}$
- If $I \subset J$ and $J \in \mathcal{I}$ then $I \in \mathcal{I}$
- If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists $j \in J - I$ such that $I \cup j \in \mathcal{I}$.

A <u>basis</u> is a maximal independent set.
The collection $\mathcal{B}$ of bases determines M

<u>Ex 1.</u>

$E = \{a,b,c,d,e,f\}$

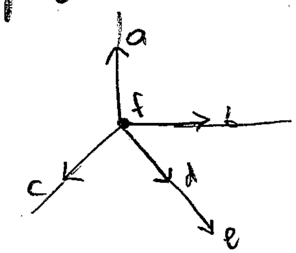
$\mathcal{I} = \{\emptyset, a, b, c, d, e, ab, ac, ad, ae, bc, bd, be, cd, ce, abc, abd, abe, acd, ace\}$

$\mathcal{B} = \{abc, abd, abe, acd, ace\}$

<u>Ex 2</u> (Linear matroids)

E = set of vectors in a vector space

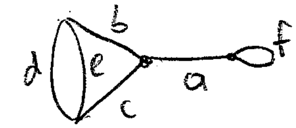
$\mathcal{I}$ = linearly independent subsets.



<u>Ex 3</u> (Graphical matroids)

E = edges of a graph

$\mathcal{I}$ = subsets of E with no cycles



<u>Ex 4</u> (Algebraic matroids)

E = elements of a field extension of $\mathbb{F}$

$\mathcal{I}$ = subsets of E algebraically independent over $\mathbb{F}$.

$\mathbb{F} \subset \mathbb{F}(x, y, z)$

$a = x+y+z$	$d = xy$
$b = x+y$	$e = x^2y^2$
$c = x-y$	$f = 1$

Many other examples!

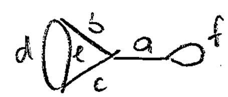
If $M=(E,\mathcal{B})$ is a matroid
$e \in E$
then

The <u>deletion</u> $M\backslash e$ has
- ground set: $E-e$
- bases: $\{B\in\mathcal{B} \mid e\notin B\} = \mathcal{B}\backslash e$

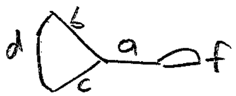
The <u>contraction</u> M/e has
- ground set: $E-e$
- bases: $\{B-e \mid B\in\mathcal{B},\ e\in B\} = \mathcal{B}/e$

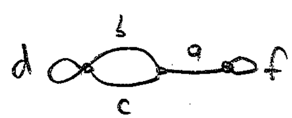
Ex

M: $\mathcal{B}=\{abc, abd, abe, acd, ace\}$



$\mathcal{B}\backslash e=\{abc, abd, acd\}$ $\qquad$ $\mathcal{B}/e=\{ab, ac\}$





If $A=\{a_1,\dots,a_k\}$ then

$M/A = M/a_1/a_2/\cdots/a_k$

$M\backslash A = M\backslash a_1\backslash a_2\backslash\cdots\backslash a_k$ $\qquad$ $M|A = M\backslash(E-A)$

Fact. M/A, $M|A$ are matroids

If $M=(E,\mathcal{B})$, $M'=(E',\mathcal{B}')$ are matroids
the <u>direct sum</u> $M\oplus M'$ has
- ground set $E\sqcup E'$
- bases: $\{B\cup B' : B\in\mathcal{B},\ B'\in\mathcal{B}'\}$

Fact: $M\oplus M'$ is a matroid.

<u>Prop</u>. The product
$$M\cdot M' := M\oplus M'$$
and coproduct
$$\Delta(M) = \sum_{A\subseteq E} (M|A)\otimes(M/A)$$
give a Hopf algebra of matroids.

For graphical matroids, this is essentially the same as the (second) Hopf algebra of graphs.