So basis elts of $H(\mathcal{G})$ are

$\cong$ classes of distributive lattices $\quad L$

$\Updownarrow \qquad\qquad \downarrow$

$\cong$ classes of posets $\quad P$

Product: $L_1 \circ L_2 = L_1 \times L_2$

$\downarrow$

$P_1 \cdot P_2 = P_1 \uplus P_2 \qquad L_1 = J(P_1),\ L_2 = J(P_2)$

Coproduct: $\Delta(L) = \sum_{x \in L} [\hat{0}, x] \otimes [x, \hat{1}]$

$\downarrow$

$\Delta(P) = \sum_{D \text{ downset}} D \otimes (P \setminus D) \qquad L \cong J(P)$

$[\hat{0}, x] \cong J(D)$

$[x, \hat{1}] \cong J(P \setminus D)$

Antipode: $S(L) = \sum_{\hat{0} = x_0 < x_1 < \dots < x_n = \hat{1}} (-1)^n [x_0, x_1] \times \dots \times [x_{n-1}, x_n]$

$\downarrow$

$S(P) = \sum_{\substack{f: P \to [n] \\ \text{surjective,} \\ \text{order preserving}}} (-1)^n \left[f^{-1}(1) \uplus \dots \uplus f^{-1}(n) \right]$

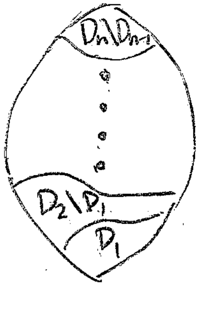
$\{x_i\}$ in $J(P)$

$\downarrow$

$\{D_i\}$ downsets in P

$\downarrow$

$f: P \to [n] \qquad f(D_i \setminus D_{i-1}) = i$



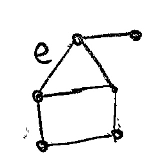
P

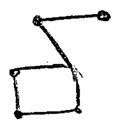
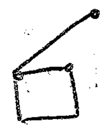
Ex 7 (Another Hopf alg. of graphs)

Let $G = (V, E)$ be a simple graph and $e \in E$

Deletion $G \setminus e$: delete edge e
keep the same vertices

Contraction G/e: delete all edges from u to v $\quad (e = u - v)$
identify vertices u, v
remove repetitions of edges

$G =$ 

$G \setminus e =$ 
$G/e =$ 

Easy lemmas: $(G \setminus e) \setminus f = (G \setminus f) \setminus e \qquad (G \setminus e)/f = (G/f) \setminus e$

$(G/e)/f = (G/f)/e \qquad (e \neq f)$

This allows us to define, for $S \subset E$, the deletion $G \setminus S$ and the contraction G/S

$G \setminus S = G \setminus s_1 \setminus \dots \setminus s_k$ for $S = \{s_1, \dots, s_k\}$

$G/S = G/s_1/\dots/s_k$

Also for $S \subset T \subset E$, let

$$G[S,T] = G \setminus (E - T) / S$$

Equivalent to deletion:

restriction $G|A = G \setminus (E - A)$

Note: If $C =$  is (the edge set of) a cycle

then $G/(C-e) = G/C$.

I.e: if you're going to contract a set A of edges you might as well contract all edges that close a cycle with A.

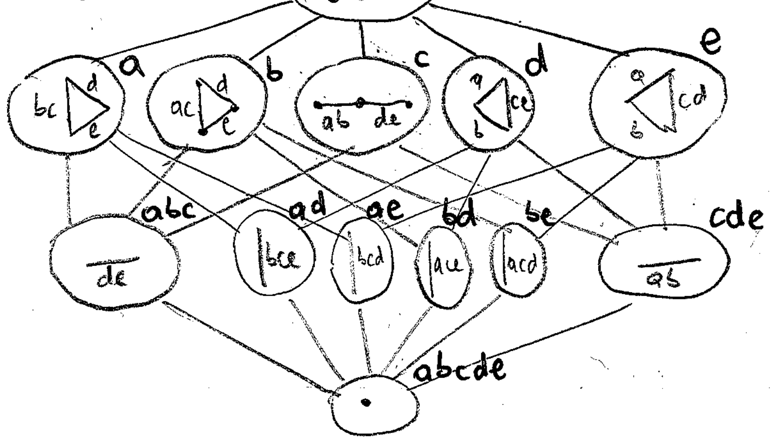
Def A $\boxed{\text{flat}}$ of G is a set A of edges such that:

If $\begin{pmatrix} \circ\ C \text{ is a cycle} \\ \circ\ e \in C \\ \circ\ C \setminus e \subseteq A \end{pmatrix}$ then $C \subseteq A$

> **Lemma**: There is a bijection between the flats of G and the contractions of G

Ex: $G =$ 

$(L_G)^{opp} =$



The poset of contractions, ordered by opposite successive contraction, is the same as the poset of flats ordered by containment. It is called the $\boxed{\text{lattice of flats}}$ L_G of G.

> **Prop** L_G is a ranked lattice

Pf Note that

F, G flat $\Rightarrow F \cap G$ flat

So we have $F \wedge G = F \cap G$.

Any finite poset with $\wedge$ and $\hat{1}$ has $\vee$:

$$F \vee G = \bigwedge_{\substack{H \geq F \\ H \geq G}} H$$

so it is a lattice.

The rank of a flat is

$$r(F) = \begin{pmatrix} \#\text{ of vertices} \\ \text{of } F \end{pmatrix} - \begin{pmatrix} \#\text{ of connected} \\ \text{components of } F \end{pmatrix}$$

(Check, easy.)

> **Prop** If F is a flat of G then in L_G
>
> $[\hat{0}, F] \cong L_{G|F}$ $\quad [F, \hat{1}] \cong L_{G/F}$
>
> If $S \subset T$ are flats, then
>
> $[S, T] \cong L_{G[S,T]}$

Pf Easy from the defs.

Prop If G, H are graphs then
$$L_G \times L_H \cong L_{G \uplus H}$$

Pf Clear.

Therefore,

$\mathcal{L} = \{$lattices of flats of simple graphs$\}$

is a hereditary family, and gives an incidence Hopf algebra.

In terms of graphs,

$\mathcal{G} = \{$finite simple graphs$\}$ $\sim$ = (isomorphism, modulo isolated vertices)

(or a family of graphs closed under deletion, contraction, disjoint union — "hereditary family")

$H(\mathcal{G}) = \text{span}(\mathcal{G})$

product: $[G]\cdot[H] = [G \uplus H]$

coproduct: $$\Delta([G]) = \sum_{F \text{ flat}} [G|F] \otimes [G/F]$$

Unit: $u(1) = \emptyset$

counit: $\epsilon(G) = \begin{cases} 1 & G = \cdots \\ 0 & \text{otherwise} \end{cases}$

is a Hopf algebra.

Ex 8 Faà di Bruno Hopf algebra

Let $\mathcal{K} = \{$disjoint unions of complete graphs$\}$

This is a hereditary family of graphs.

Let $x_n = [K_{n+1}] = [$  $]$ for $n \geq 1$.

n+1 vertices
all edges

$\mathcal{F} := H(\mathcal{K})$ is generated as an algebra by $x_1, x_2, x_3, \ldots$ with free commutative product.

$$\Delta(x_n) = \sum_{k=1}^{n} B_{n+1,k+1}(1, x_1, x_2, \ldots) \otimes x_k$$

where $B_{n,k}(y_1, y_2, \ldots)$ are the <u>partial Bell polynomials</u>:

$$B_{n,k}(y_1, y_2, \ldots) = \sum b_{j_1 j_2 \ldots} y_1^{j_1} y_2^{j_2} \cdots$$

$$\left(B_{6,3}(y_1, y_2, \ldots) = \underset{(6=2+2+2)}{15 y_2^3} + \underset{(6=3+2+1)}{60 y_3 y_2 y_1} + \underset{(6=4+1+1)}{15 y_4 y_1^2} \right)$$

where $b_{j_1 j_2 \ldots}$ = # of ways of partitioning $[n]$ into k groups: j_1 of size 1, j_2 of size 2, ...

$$= \binom{n}{\underbrace{1, \ldots, 1}_{j_1}, \underbrace{2, \ldots, 2}_{j_2}, \ldots} \cdot \frac{1}{j_1!} \cdot \frac{1}{j_2!} \cdots$$

So $$B_{n,k}(y_1, y_2, \ldots) = \sum_{\substack{j_1 + j_2 + \cdots = k \\ j_1 + 2j_2 + 3j_3 + \cdots = n}} \frac{n!}{j_1! \, j_2! \cdots} \left(\frac{y_1}{1!}\right)^{j_1} \left(\frac{y_2}{2!}\right)^{j_2} \cdots$$

More cleanly,

$$e^{\left(\sum_{m\geq 1} Y_m \frac{t^m}{m!}\right)X} = \sum_{n,k} B_{n,k}(Y_1, Y_2, \dots) \frac{t^n}{n!} \frac{X^k}{k!}$$

There is much to say about these polynomials.

Thm (Faà di Bruno formula)

$$[f(g(x))]^{(n)} = \sum_{k=1}^{n} f^{(k)}(g(x))\, B_{n,k}(g'(x), g''(x), \dots)$$

Pf. Exercise. (E.g., by induction)

Thm (Lagrange Inversion Formula)

Let $f(x) = \sum_{n\geq 1} f_{n-1} \frac{t^n}{n!}$, $g(x) = \sum_{n\geq 1} g_{n-1} \frac{t^n}{n!}$ $[f_0 = g_0 = 1]$

be compositional inverses. Then

$$g_n = \sum_{k\geq 1} (-1)^k B_{n+k,k}(0, f_1, f_2, \dots)$$

Ex: $y = x + ax^2 + bx^3 + cx^4 + \dots \rightarrow x = y + Ay^2 + By^3 + Cy^4 + \dots$

where
$A = -a$
$B = 2a^2 - b$
$C = 5ab - c - 5a^3$
$D = 6ac + 3b^2 + 14a^4 - d - 21a^2 b$
$\vdots$

$$4!\,C = g_3 = -B_{4,1}(0, f_1, \dots) + B_{5,2}(0, f_1, \dots) - B_{6,3}(0, f_1, \dots) + 0$$

$$= -\underbrace{f_3}_{4=4} + \underbrace{(10 f_2 f_1 + 5 f_3 \cdot 0)}_{5=3+2=4+1} - \underbrace{(15 f_1^3 + 0 + 0)}_{6=2+2+2=1+\dots} + 0 - \dots$$

$$24C = -4!\,c + 10(3!\,b)(2!\,a) - 15(2!\,a)^3 = -24c + 120bc - 120a^3$$

Lemma (Faà di Bruno formula, v.2)

If $f(x) = \sum_{n\geq 1} a_n \frac{x^n}{n!}$, $g(x) = \sum_{n\geq 1} b_n \frac{x^n}{n!}$ then

$$f(g(x)) = \sum_{n\geq 1} \sum_{k\geq 1} a_k B_{n,k}(b_1, b_2, \dots) \frac{x^n}{n!}$$

Pf (From v.1)

Compare the coeffs of x^n in both sides.

LHS: $f(g(x))^{(n)}\big|_{x=0}$

RHS: $\sum_{k\geq 1} a_k B_{n,k}(b_1, b_2, \dots)$

$$= \sum_{k\geq 1} f^{(k)}(\underbrace{0}_{=g(0)})\, B_{n,k}(g'(0), g''(0), \dots)$$

They are equal by v.1 of the formula ∎

We'll see that composition of such "divided power series" is closely related to the Faà di Bruno Hopf algebra.

Let $D = \{$"divided power series" $t + d_2 \frac{t^2}{2!} + d_3 \frac{t^3}{3!} + \dots : d_i \in k\}$ considered as a group under composition.

The group of characters of a Hopf algebra

A character of a Hopf algebra H is an algebra map $\chi : H \to \mathbb{K}$; so

$$\chi(a \pm b) = \chi(a) \pm \chi(b)$$
$$\chi(ab) = \chi(a)\chi(b)$$
$$\chi(1) = 1.$$

Prop The set of characters $\mathbb{X}(H)$ is a group under convolution.

Pf

- If χ_1, χ_2 are characters then $\chi_1 * \chi_2$ is:

$$(\chi_1 * \chi_2)(ab) = \sum_{(ab)} \chi_1((ab)_{(1)})\,\chi_2((ab)_{(2)})$$

Δ multip.

$$= \sum_{(a),(b)} \chi_1(a_{(1)}b_{(1)})\,\chi_2(a_{(2)}b_{(2)})$$

$$= \sum_{(a),(b)} \chi_1(a_{(1)})\chi_1(b_{(1)})\chi_2(a_{(2)})\chi_2(b_{(2)})$$

$$= \Big(\sum_{(a)} \chi_1(a_{(1)})\chi_2(a_{(2)})\Big)\Big(\sum_{(b)} \chi_1(b_{(1)})\chi_2(b_{(2)})\Big)$$

$$= (\chi_1 * \chi_2)(a)\,(\chi_1 * \chi_2)(b)$$

The others are easier.

- Identity: Counit ϵ

$$\sum_{(a)} \chi((a)_{(1)})\,\epsilon(a_{(2)}) = \chi\Big(\sum_{(a)} a_{(1)}\,\epsilon(a_{(2)})\Big) = \chi(a)$$

- Inverse: the inverse of χ is $\chi \circ S$:

$$\sum_{(a)} \chi(a_{(1)})\,\chi(S(a_{(2)})) = \chi\Big(\sum_{a} a_{(1)}\, S(a_{(2)})\Big) \overset{I*S = u\epsilon}{=} \chi(\epsilon(a)) = \epsilon(a) \qquad \blacksquare$$

Prop (Doubilet-Rota-Stanley) The group of characters $\mathbb{X}(\mathcal{F})$ of the Faà di Bruno Hopf algebra is antiisomorphic to the group of divided power series D under composition (Need char $\mathbb{K} = 0$)

Pf. Define $F : \mathbb{X}(\mathcal{F}) \longrightarrow D$

$$f \longmapsto \sum_{n \geq 1} f(x_{n-1}) \frac{t^n}{n!} =: F(f)$$

This is a bijection. (f determined by $f(x_0), f(x_1), \ldots$). Now:

$$F(f * g) = \sum_{n \geq 1} (f * g)(x_{n-1}) \frac{t^n}{n!}$$

$$= \sum_{n \geq 1} \sum_{k=1}^{n} f(B_{n,k}(1, x_1, x_2, \ldots))\, g(x_{k-1}) \frac{t^n}{n!}$$

$$= \sum_{n \geq 1} \sum_{k \geq 1} B_{n,k}(f(1), f(x_1), f(x_2), \ldots)\, g(x_{k-1}) \frac{t^n}{n!}$$

Faà di Bruno

$$= F(g) \circ F(f)$$

So F preserves the product, but backwards.

$\Rightarrow F$ is a group antiisomorphism $\blacksquare$

Corollary

The antipode of the Faà di Bruno Hopf alg is

$$S(x_n) = \sum_{k\geq 1} (-1)^k \mathbf{B}_{n+k,k}(0, x_1, x_2, \dots)$$

Pf

Compute (twice) the inverse of $F(f) = \sum_{n\geq 1} f(x_{n-1}) \frac{t^n}{n!}$.

- It is $F(\underset{\uparrow \atop \text{in } \mathbb{X}(\mathcal{F})}{f^{-1}}) = F(f \circ S) = \sum_{n\geq 1} f(S(x_{n-1})) \frac{t^n}{n!}$.

- It is $\sum_{n\geq 1} g_{n-1} \frac{x^n}{n!}$ where, by Lagrange inversion,

$$g_n = \sum_{k\geq 1} (-1)^k \mathbf{B}_{n+k,k}(0, f(x_1), f(x_2), \dots)$$

$$= f\Big(\sum_{k\geq 1} (-1)^k \mathbf{B}_{n+k,k}(0, x_1, x_2, \dots)\Big)$$

Compare coeffs of $\frac{t^{n+1}}{(n+1)!}$. They are equal for all f, so the result holds. ∎

So $\mathcal{F}$ is an algebraic encoding of composition of $\sum_{n\geq 1} a_n \frac{t^n}{n!}$:

$$\begin{pmatrix}\text{formula for} \\ \text{antipode of } \mathcal{F}\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{formula} \\ \text{for } f^{-1}(x)\end{pmatrix}$$

$$\begin{pmatrix}\text{formula for} \\ \text{coproduct of } \mathcal{F}\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{formula for} \\ f(g(x))\end{pmatrix}$$

([Haiman-Schmitt] prove the formula for $S(x_n)$ combinatorially, and use it to deduce Lagrange inversion.)