Ex 5 (Incidence Hopf alg. of lattices)

A *lattice* is a poset L such that:

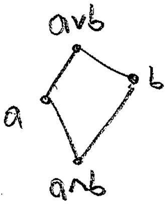
any two elements $a, b \in L$ have a *join* $a \vee b$ and a *meet* $a \wedge b$ such that

- $a \vee b \geq a$
  $a \vee b \geq b$
  If $c \geq a, c \geq b$
  then $c \geq a \vee b$

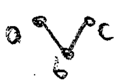
  least upper bound

- $a \wedge b \leq a$
  $a \wedge b \leq b$
  If $d \leq a, d \leq b$
  then $d \leq a \wedge b$.

  greatest lower bound

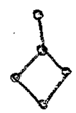


Examples:

no: 

$a \vee c = ?$

$b \vee c = ?$

yes: 

Other examples:

- $(2^S, \subseteq)$ $\quad A \wedge B = A \cap B, \; A \vee B = A \cup B$
- $(\mathbb{N}, \text{divisibility})$ $\quad a \wedge b = \gcd(a,b) \quad a \vee b = \operatorname{lcm}(a,b)$
- $(\text{subgroups of } G, \subseteq)$ $\quad H \wedge H' = H \cap H' \quad H \vee H' = \langle H, H' \rangle$

Easy facts/exercises

- Every lattice has a $\hat{0}$ and $\hat{1}$.
- Every interval of a lattice is a lattice.
- Every product of lattices is a lattice.
- $\vee$ and $\wedge$ are commutative and associative.

So there is an incidence Hopf algebra of lattices (say $\sim$ = isomorphism), but maybe lattices are too general to say anything new about it.

Ex 6. (Another Hopf algebra of posets)

A lattice L is *distributive* if $\wedge$ and $\vee$ are:

$$\begin{cases} a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \end{cases} \quad \forall a,b,c \in L$$

Easy facts/exercises

- Either property implies the other one
- Distributive lattices are closed under taking subintervals, products.

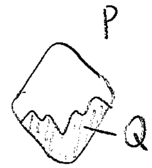
$\Rightarrow$ There is an incidence Hopf alg of distrib lattices.

Why is the incidence Hopf alg of distributive lattices nice to have? Because distributive lattices have a lot of structure:

Let P be a poset and

$J(P) = \{\underline{\text{downsets}} \text{ of } P\}$

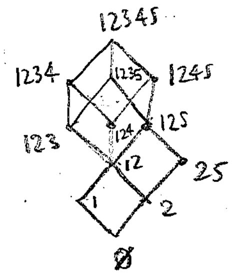
$\hookrightarrow Q \subseteq P$ such that $x \leq y,\ y \in Q \Rightarrow x \in Q$



Then $(J(P), \subseteq)$ is distributive $\left(\begin{array}{c} A,B \in J(P) \\ \Downarrow \\ A \cup B, A \cap B \in J(P) \end{array}\right)$

$A \vee B = A \cup B, \quad A \wedge B = A \cap B$

Ex: $P =$ 

$J(P) =$ 

Fundamental Theorem for Finite Distributive Lattices

Let L be a finite distributive lattice. There is a unique (up to $\cong$) poset P such that

$$L \cong J(P)$$

(Birkhoff, 1947)

Sketch of Proof:

Existence:

Say $p \in L$ is join-irreducible if there are no $a, b \in L$ such that $a \vee b = p$.

Let $P = \{\text{join-irred elts of } L\}$

Let P inherit the partial order from L.

Claim: $L \cong J(P)$

Maps: $\phi : L \longrightarrow J(P)$

$t \mapsto f(t) = \{s \in P : s \leq t\}$

$L \longleftarrow J(P) : \phi^{-1}$

$\bigvee_{i \in I} i \longmapsfrom I$

(Check details.)

Uniqueness:

Claim: The poset of join-irreducibles of $J(P)$ is isomorphic to P

Pf: There is a bijection $\{\text{downsets}\} \longleftrightarrow \{\text{antichains}\}$

$D \longmapsto D_{max} = \{\text{maxl elts}\}$

$P_{\leq A} = \{p : p \leq a \text{ for some } a \in A\} \longmapsfrom A$

So the join-ireds of $J(P)$ are those D where $|D_{max}| = 1$, i.e. the sets $P_{\leq p}$, $p \in P$. ∎

Then $J(P) \cong J(Q) \Rightarrow P \cong Q$

So basis elts of $H(\mathcal{J})$ are

$$\begin{array}{ccc} \cong \text{ classes of distributive lattices} & & L \\ \Updownarrow & & \downarrow \\ \cong \text{ classes of posets} & & P \end{array}$$

Product: $L_1 \circ L_2 = L_1 \times L_2$

$$\downarrow$$

$P_1 \cdot P_2 = P_1 \uplus P_2$ $\qquad L_1 = J(P_1),\ L_2 = J(P_2)$

Coproduct: $\Delta(L) = \sum_{x \in L} [\hat{0}, x] \otimes [x, \hat{1}]$

$$\downarrow$$

$\Delta(P) = \sum_{D \text{ downset}} D \otimes (P \setminus D)$ $\qquad L \cong J(P)$

$[\hat{0}, x] \cong J(D)$

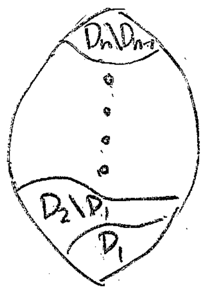
$[x, \hat{1}] \cong J(P \setminus D)$

Antipode: $S(L) = \sum_{\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}} (-1)^n [x_0, x_1] \times \cdots \times [x_{n-1}, x_n]$

$$\downarrow$$

$S(P) = \sum_{f: P \to [n]} (-1)^n \left[f^{-1}(1) \uplus \cdots \uplus f^{-1}(n)\right]$

surjective, order preserving



P

$$\left(\begin{array}{l} \{x_i\} \text{ in } J(P) \\ \downarrow \\ \{D_i\} \text{ downsets in } P \\ \downarrow \\ f: P \to [n] \quad f(D_i \setminus D_{i-1}) = i. \end{array}\right)$$