

Ideals, Quotients

- If B is a bialgebra, a biideal $I \subset B$ is a subset which is a two-sided ideal and a two-sided coideal.

In that case, B/I is a quotient bialgebra, which inherits the bialg. structure from B .
The 1st from Thm holds.

- If H is a Hopf algebra, a Hopf ideal is a biideal I such that $S(I) \subset I$.

In that case, H/I is a quotient Hopf algebra which inherits its structure from H .

The 1st from Thm holds.

Dual

- If $(H, m, u, \Delta, \epsilon, S)$ is a finite-dimensional Hopf algebra, then $(H^*, \Delta^*, \epsilon^*, m^*, u^*, S^*)$ is the dual Hopf algebra.

Often the antipode comes for free.

Def A bialgebra H is graded if

$$H = \bigoplus_{n \geq 0} H_n$$

$$H_i H_j \subseteq H_{i+j} \quad \forall i, j \geq 0$$

$$\Delta H_n \subseteq \bigoplus_{i+j=n} (H_i \otimes H_j) \quad \forall n \geq 0$$

$$\epsilon H_n = 0 \quad \forall n \geq 1$$

It is connected if $H_0 \cong \mathbb{K}$

Ex.

$$\mathbb{K}\langle x \rangle = \bigoplus_{n \geq 0} \{x^n\}, \quad x^i x^j = x^{i+j}, \quad \Delta x^n = \sum_{i+j=n} x^i \otimes x^j$$

- $\mathbb{K}\{\text{isom. classes of finite graphs}\} = H$

$H_n = \text{graphs on } n \text{ vertices}$

$$G_1 \cdot G_2 = \text{disjoint union}$$

$$\Delta(G) = \sum_{s \subset V} G|_s \otimes G|_{V-s}$$

HW: Check this is a bialgebra

- $\mathbb{K}\{\text{perm. of some } [n]\} = H$

$$H_n = \mathbb{K}S_n$$

$$\pi_1 \cdot \pi_2 = \sum \text{shuffles of } \pi_1, \pi_2$$

$$\Delta(\pi) = \sum_{i \subset [n]} \text{st}(\pi|_i) \otimes \text{st}(\pi|_{[n]-i})$$

Proposition (Takeuchi '71)

A graded, connected bialgebra H has an antipode. If $\pi = I - u \in \text{Hom}(H, H)$ then

$$S = \sum_{n \geq 0} (-1)^n m^{n-1} \pi^{\otimes n} \Delta^{n-1}$$

which turns it into a Hopf algebra

Convention: $m^0 = \Delta^0 = \text{id}$ Note $\Delta^n(H_m) = 0$ for $n > m$

$$m^{-1} = u, \Delta^{-1} = \epsilon$$

So $\Delta^n(h)$ is a finite sum for any $h \in H$.

PF Recall that in the convolution product

$$\begin{aligned} \pi^{\times n} &= \sum_{(h)} \pi(h_{(1)}) \cdots \pi(h_{(n)}) \\ &= m^{n-1} \pi^{\otimes n} \Delta^{n-1} \end{aligned}$$

so really

$$S = \sum_{n \geq 0} (-1)^n \pi^{\times n}$$

-Then

$$S * I = \left[\sum_{n \geq 0} (-1)^n \pi^{\times n} \right] * (\pi + u \epsilon)$$

$$= \sum_{n \geq 0} (-1)^n \pi^{\times (n+1)} + \sum_{n \geq 0} (-1)^n \pi^{\times n}$$

$$= \pi^{\times 0} = u \epsilon$$

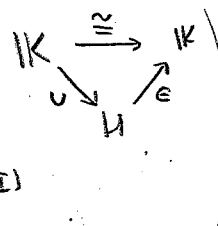
Similarly $I * S = u \epsilon$. \square

In fact, note that this applies for any co-nilpotent bialgebra, where any $h \in H$ satisfies $\Delta^n h = 0$ for some $n \geq 0$.

Remark

- A graded bialgebra H is connected $\iff u \epsilon|_{H_0} = I|_{H_0}$

(This is because we always have
 • if $H_0 \cong \mathbb{K}$ then $u = \epsilon^{-1}$
 • if $\dim H_0 > 1$, $\dim(\text{Im } u \epsilon) = 1 < \dim(\text{Im } I)$)



- On H_n ($n \geq 1$), $u \epsilon = 0$

So $(I - u \epsilon)(h)$ just drops the H_0 part of h .

This formula isn't always so practical.