Ideals, Quotients

- If B is a bialgebra, a <u>biideal</u> $I \subset B$ is a subset which is a two-sided ideal and a two-sided coideal.

  In that case, B/I is a <u>quotient bialgebra</u>, which inherits the bialg. structure from B.
  The 1st Isom Thm holds.

- If H is a Hopf algebra, a <u>Hopf ideal</u> is a biideal I such that $S(I) \subset I$.
  In that case, H/I is a <u>quotient Hopf algebra</u> which inherits its structure from H.
  The 1st Isom Thm holds.

Dual

- If $(H, m, u, \Delta, \epsilon, S)$ is a finite-dimensional Hopf algebra, then $(H^*, \Delta^*, \epsilon^*, m^*, u^*, S^*)$ is the <u>dual Hopf algebra</u>.

Often the antipode comes for free.

Def A bialgebra H is <u>graded</u> if

- $H = \bigoplus_{n \geq 0} H_n$
- $H_i H_j \subseteq H_{i+j} \quad \forall i,j \geq 0$
- $\Delta H_n \subseteq \bigoplus_{i+j=n} (H_i \otimes H_j) \quad \forall n \geq 0$
- $\epsilon H_n = 0 \quad \forall n \geq 1$

It is <u>Connected</u> if $H_0 \cong \mathbb{K}$

<u>Ex.</u>

- $\mathbb{K}[x] = \bigoplus_{n \geq 0} \{x^n\}, \quad x^i x^j = x^{i+j}, \quad \Delta x^n = \sum_{i+j} x^i \otimes x^j$

- $\mathbb{K}\{\text{isom. classes of finite graphs}\} = H$
  $H_n =$ graphs on n vertices

  $G_1 \cdot G_2 =$ disjoint union
  $\Delta(G) = \sum_{S \subset V} G|_S \otimes G|_{V-S}$

  HW: Check this is a bialgebra

- $\mathbb{K}\{\text{perms. of some } [n]\} = H$
  $H_n = \mathbb{K}S_n$

  $\pi_1 \cdot \pi_2 = \sum$ shuffles of π_1, π_2
  $\Delta(\pi) = \sum_{\text{cuts}} st(\pi_1 \ldots \pi_i) \otimes st(\pi_{i+1} \ldots \pi_n)$

Theorem (Takeuchi '71)

A graded, connected bialgebra H has an antipode. If $\pi = I - u\epsilon : H \to H$ then

$$S = \sum_{n \geq 0} (-1)^n m^{n-1} \pi^{\otimes n} \Delta^{n-1}$$

which turns it into a Hopf algebra

Convention: $m^0 = \Delta^0 = id$

$m^{-1} = u, \Delta^{-1} = \epsilon$

Note $\Delta^n(H_m) = 0$ for $n > m$

So $\Delta^n(h)$ is a finite sum for any $h \in H$.

Pf Recall that in the convolution product

$$\pi^{*n} = \sum_{(h)} \pi(h_{(1)}) \cdots \pi(h_{(n)})$$

$$= m^{n-1} \pi^{\otimes n} \Delta^{n-1}$$

so really

$$S = \sum_{n \geq 0} (-1)^n \pi^{*n}$$

Then

$$S * I = \left[\sum_{n \geq 0} (-1)^n \pi^{*n}\right] * (\pi + u\epsilon)$$

$u\epsilon$ is 1 for convolution

$$= \sum_{n \geq 0} (-1)^n \pi^{*(n+1)} + \sum_{n \geq 0} (-1)^n \pi^{*n}$$

$$= \pi^{*0} = u\epsilon$$

Similarly $I * S = u\epsilon$. ■

In fact, note that this applies for any conilpotent bialgebra, where any $h \in H$ satisfies $\Delta^n h = 0$ for some $n \geq 0$.

Remark

- A graded bialgebra H is connected $\iff$ $u\epsilon|_{H_0} = I|_{H_0}$

(This is because we always have

$$\begin{tikzcd} \mathbb{K} \arrow[rr, "\cong"] \arrow[dr, "u"'] & & \mathbb{K} \\ & H \arrow[ur, "\epsilon"'] & \end{tikzcd}$$

- if $H_0 \cong \mathbb{K}$ then $u = \epsilon^{-1}$
- if $\dim H_0 > 1$, $\dim(\operatorname{Im} u\epsilon) = 1 < \dim(\operatorname{im} I)$)

- On H_n $(n \geq 1)$, $u\epsilon = 0$

So $(I - u\epsilon)(h)$ just drops the H_0 part of h.

This formula isn't always so practical.