2. (a) To prove this identity, we equate coefficients on the left- and right-hand sides of the equation. By definition of p_n,

$$p_n(x+y) = \sum_{k\geq 0} S(n,k)(x+y)^k$$

Applying the binomial theorem to $(x+y)^k$, we have

$$p_n(x+y) = \sum_{k\geq 0} S(n,k) \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j}$$

Let $r, s \geq 0$. Then

$$[x^r y^s] \sum_{k\geq 0} S(n,k) \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j} = S(n, r+s)\binom{r+s}{r}$$

i.e.,

$$[x^r y^s] p_n(x+y) = S(n, r+s)\binom{r+s}{r}$$

As for the right side: First, to avoid confusion in the argument below, we replace the variable k in this expression with j. That is, the right-hand side is

$$\sum_{j=0}^{n} \binom{n}{j} p_j(x) p_{n-j}(y)$$

Writing

$$p_j(x) = \sum_{\ell \geq 0} S(j,\ell)x^\ell, \qquad p_{n-j}(y) = \sum_{m \geq 0} S(n-j,m)y^m$$

we have that

$$\sum_{j=0}^{n} \binom{n}{j} p_j(x)p_{n-j}(y) = \sum_{j=0}^{n} \binom{n}{j} \left(\sum_{\ell \geq 0} S(j,\ell)x^\ell \right) \left(\sum_{m \geq 0} S(n-j,m)y^m \right)$$

For $r, s \geq 0$,

$$[x^r y^s] \left(\sum_{\ell \geq 0} S(j,\ell)x^\ell \right) \left(\sum_{m \geq 0} S(n-j,m)y^m \right) = S(j,r)S(n-j,s)$$

and it follows that

$$[x^r y^s] \sum_{j=0}^{n} \binom{n}{j} p_j(x)p_{n-j}(y) = \sum_{j=0}^{n} \binom{n}{j} S(j,r)S(n-j,s)$$

Hence it remains to show that

$$S(n, r+s)\binom{r+s}{r} = \sum_{j=0}^{n} \binom{n}{j} S(j,r)S(n-j,s)$$

This may be done via a combinatorial argument. Indeed, both expressions count the number of ways to decorate a subset of $[n]$ and then to partition $[n]$ such that each block contains only decorated elements or only undecorated elements, and there are exactly r blocks and exactly s blocks containing undecorated elements. On the left-hand side, we first choose a partition of $[n]$ into $r+s$ blocks ($S(n, r+s)$ ways) and then choose r of these $r+s$ blocks to have their elements decorated ($\binom{r+s}{r}$ ways). On the right-hand side, we condition on the number $1 \leq j \leq n$ of decorated elements. For each $1 \leq j \leq n$, we first choose j elements to be decorated ($\binom{n}{j}$ ways), then partition these j decorated elements into r blocks ($S(j,r)$ ways) and partition the remaining $n-j$ undecorated elements into s blocks ($S(n-j,s)$ ways). Summing over all j yields the desired equality.