5. *(talked with Nina and Emily) Prove that the number of partitions of n in which no part appears exactly once equals the number of partitions of n into parts not congruent to $\pm($* mod 6).

We know that the the generating function for the number of partitions of n is given by

$$\sum_{n=1}^{\infty} p(n)x^n = \prod_{i=1}^{\infty}(1 + x^i + x^2 i + \cdots) = \prod_{i=1}^{\infty} \frac{1}{1 - x^i}$$

However, if from one of the sums being multiplied together, we chose x^i, (the second element in the sum) this amounts to choosing a partition of n where i only appears once. Therefore the generating function for partitions of n where no element appears exactly once, is

$$\begin{aligned}\sum_{n=1}^{\infty} q(n)x^n &= \prod_{i=1}^{\infty} \frac{1}{1 - x^i} - x^i \\ &= \prod_{i=1}^{\infty} \frac{1 - x^i + x^{2i}}{1 - x^i} \\ &= \prod_{i=1}^{\infty} \frac{\frac{1+x^{3i}}{1+x^i}}{1 - x^i} \\ &= \prod_{i=1}^{\infty} \frac{1 + x^{3i}}{1 - x^{2i}}\end{aligned}$$

Now rewriting this last product by breaing up the product of all the $\frac{1}{1-x^{2i}}$ into a product of all the even numbers $2(\mod 6)$, $4(\mod 6)$, and $0(\mod 6)$, we have

$$\begin{aligned}\prod_{i=1}^{\infty} \frac{1 + x^{3i}}{1 - x^{2i}} &= \left(\prod_{i=1}^{\infty} \frac{1}{1 - x^{6i-4}}\right)\left(\prod_{i=1}^{\infty} \frac{1}{1 - x^{6i-2}}\right)\left(\prod_{i=1}^{\infty} \frac{1}{1 - x^{6i}}\right)\left(\prod_{i=1}^{\infty} 1 + x^{3i}\right) \\ &= \left(\prod_{i=1}^{\infty} \frac{1}{1 - x^{6i-4}}\right)\left(\prod_{i=1}^{\infty} \frac{1}{1 - x^{6i-2}}\right)\left(\prod_{i=1}^{\infty} \frac{1 + x^{3i}}{1 - x^{6i}}\right) \\ &= \left(\prod_{i=1}^{\infty} \frac{1}{1 - x^{6i-4}}\right)\left(\prod_{i=1}^{\infty} \frac{1}{1 - x^{6i-2}}\right)\left(\prod_{i=1}^{\infty} \frac{1}{1 - x^{3i}}\right)\end{aligned}$$

Now examining the coefficient of x^n we see that it must have taken exponents that were either $2 \mod 6$, $4 \mod 6$, $3 \mod 6$ or $0 \mod 6$, that is the number of ways to form x^n are exactly the number of partitions into n inot parts not parts not congruent to $\pm(\mod 6)$. giving us the desired equality.