5. **(An identity for partitions)** (Idea found in the paper "Integrals, partitions and MacMahon's Theorem", by George Andrews et al.)

Let A_n =Partitions of n not containing any part exactly once and B_n = Partitions of n into parts not congruent to $\pm$ 1 (mod 6) . Let $n = \sum_{k=1}^{n} kp_k \in A_n$ where p_k represents the number of times in which the part k appears (Note that the sum does not need to reach n, since any part greater than $n/2$ could only appear once). Since this partition is in A_k, $p_k \neq 1$ and it can be decomposed as $p_k = q_k + r_k$, with $q_k \in \{0, 3\}$ and r_k an even number. This partition is unique: for any i we have $i = 0 + i$ if i is even or $i = 3 + (i - 3)$ if it is odd. Now let us build a new partition $\sum_{l=1}^{n} la_j$:

$$
\begin{aligned}
a_{6k+1} &= 0 \\
a_{6k+2} &= \tfrac{1}{2} r_{3k+1} \\
a_{6k+3} &= \tfrac{1}{3} q_{2k+1} + r_{6k+3} \\
a_{6k+4} &= \tfrac{1}{2} r_{3k+2} \\
a_{6k+5} &= 0 \\
a_{6k+6} &= \tfrac{1}{3} q_{2k+2} + r_{6k+6}
\end{aligned}
$$

From the construction it is clear that the new partition has no parts congruent to ± 1, furthermore it is in B_n because the modified coefficients exactly cancel out the change of factor l:

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 + ... = 2\frac{1}{2}r_1 + 3\frac{1}{3}(q_1 + r_3) + 4\frac{1}{2}r_2 + 6(\frac{1}{3}q_2 + r_6) + ... =$$

$$= (q_1 + r_1) + 2(q_2 + r_2) + ... = p_1 + 2p_2 + ... = n$$

then it remains to prove that it is actually a bijection. Surjectivity is clear, since the fractions before the summands in a_{6k+2}, a_{6k+3}, a_{6k+4},a_{6k+6} guarantee that this numbers can always grow in just one unity, and thus the coefficients can take any integer value, which gives all possible partitions in B_n.

Let us tackle injectivity. The terms a_{6k+2} and a_{6k+4} are uniquely determined from the terms r_{3k+1} and r_{3k+2}. A pair of partitions with different q_{3k+1} or q_{3k+2} would be sent to partitions with equal a_{6k+2} or a_{6k+4}; however, q_{3k+1} (with q_{3k+2} the argument is equivalent) can be written as $q_{2(\frac{3}{2}k)+1}$ if k is even or as $q_{2(\frac{3k-1}{2})+2}$ if k is odd; this means that q_{3k+1} will affect the coefficients $a_{6(\frac{3}{2}k)+3}$ or $a_{6(\frac{3k-1}{2}k)+6}$, but neither $r_{6(\frac{3}{2}k)+3}$ nor $r_{6(\frac{3k-1}{2}k)+6}$ can account for the difference of 1 that is introduced by the variation in q_{3k+1}, since they are even. There is no problem with the coefficients a_{6k+3} and a_{6k+6} since for a pair of different q and r the sum could not be the same due to the difference of 3 in the values of q and the evenness of r. Summarizing, one partition in B_n is obtained uniquely from one partition in A_n, hence we have a bijection.