(3) (A symmetric distribution for Dyck paths) *For a Dyck path P let $a(P)$ be the number of steps up that P takes before its first step down, and let $b(P)$ be the number of times that P returns to the x-axis after it leaves it for the first time. Prove that the statistics a and b are symmetrically distributed; that is:*

$$\sum_{P\ Dyck} x^{a(P)} y^{b(P)} = \sum_{P\ Dyck} x^{b(P)} y^{a(P)}$$

where the sum is over all Dyck paths of length $2n$.

Let

$$F(x,y) = \sum_{P \text{ Dyck}} x^{a(P)} y^{b(P)}$$

so that the statement of symmetric distribution can be summed up as $F(x,y) = F(y,x)$, or really that the coefficient of $x^j y^k$ in $F(x,y)$ is the same as the coefficient of $x^k y^j$.

We can accomplish this proof by showing that a bijection

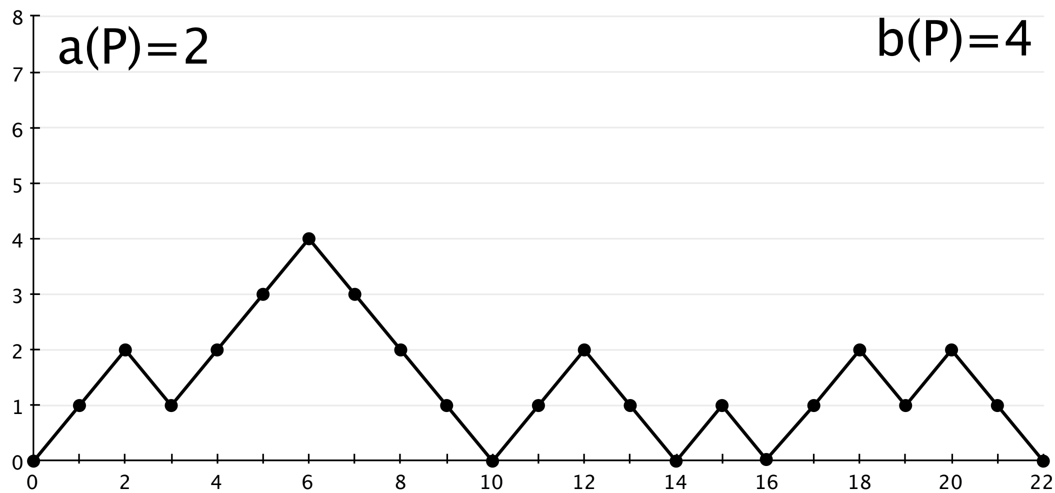
$$\{P | a(P) = j, b(P) = k\} \longleftrightarrow \{P | a(P) = k, b(P) = j\}$$

exists. I will actually show a stronger result—that a bijection

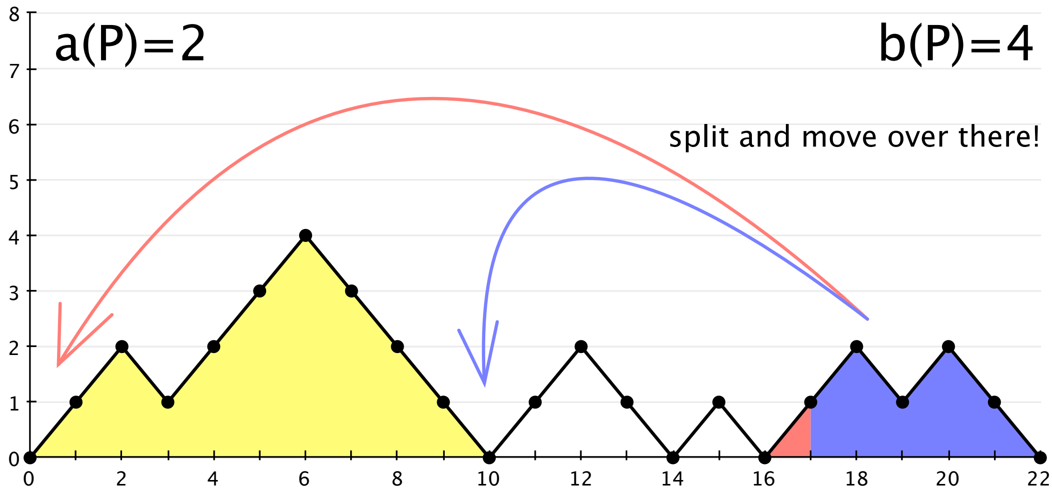
$$\varphi : \{P | a(P) = j, b(P) = k\} \longleftrightarrow \{P | a(P) = j+1, b(P) = k-1\}$$

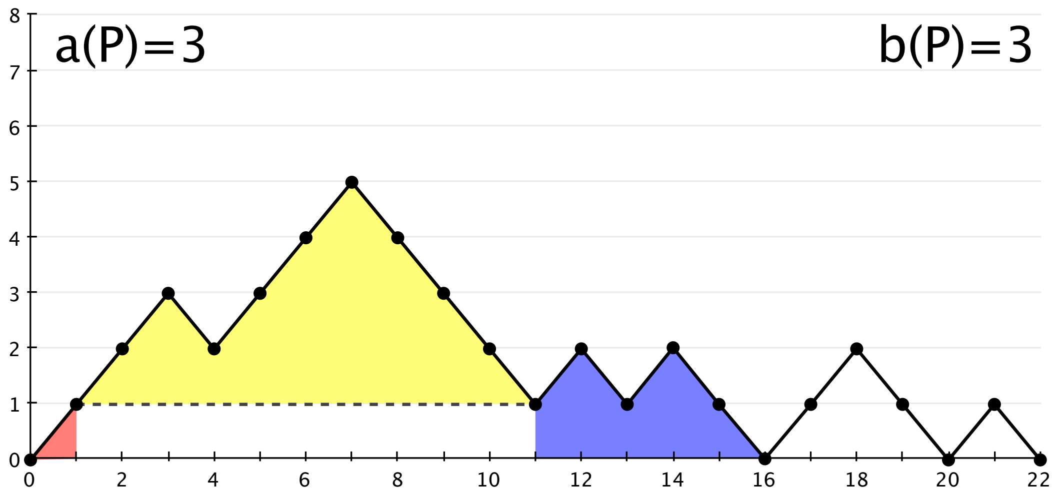
where the values make sense, i.e. $1 \le j < j+1 \le n$ and $n \ge k > k-1 \ge 1$.

The bijection φ is simple, and is best illustrated using the following example of a Dyck path where $2n = 11$, $a(P) = 2$, and $b(P) = 4$:



This Dyck path can be represented using the word "(UUDUUUDDDD)(UUDD)(UD)(UUDUDD)", where parentheses are used to emphasize the $b(P) = 4$ sub-Dyck paths.
Now, to increase $a(P)$ by 1 and decrease $b(P)$ by 1, we take the sub-Dyck path from x-coordinates 16-22 ("UUDUDD") and place the first letter (always a "U") in front of the first sub-Dyck path (shown in the image at x-coordinate 0) and place the rest of the letters ("UDUDD") after the first sub-Dyck path (in the image at x-coordiante 10). The next two figures sum up this process for creating $\varphi(P)$ for which $a(\varphi(P)) = 3$ and $b(\varphi(P)) = 3$





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The inverse φ^{-1} is just as easy. Look at the first sub-Dyck path and draw a horizontal line at a height of $y = 1$ starting at point $(1, 1)$ and moving to the right and stopping at your first intersection point, as shown by the dotted line in the previous figure. The part of the sub-Dyck path to the left and to the right of the yellow-colored region above can be placed at the end of the entire Dyck path, adding 1 to k (and no more because the blue region can only have one point touching the x-axis), and the yellow-colored region moves down, decreasing n by 1, and not affecting the value of k as it only returns to the line $y = 1$ once by construction.

So, by repeated application of either φ or φ^{-1}, we can show that there is a bijection that switches the values of $a(\cdot)$ and $b(\cdot)$. For instance, for all Dyck paths P such that $a(P) = j$ and $b(P) = k$ for some j and k, the bijection that maps these to the set of Dyck paths with these statistics switched is φ^{k-j}.