6. (Bonus problem: cycles of even and odd permutations.)

(a) Let e_n be the total number of cycles among all even permutations of $[n]$, and o_n be the total number of cycles among all odd permutations of $[n]$. Prove that

$$e_n - o_n = (-1)^n (n-2)!$$

We have the polynomial equality

$$x(x+1)\cdots(x+n-1) = \sum_{k=1}^{n} C(n,k)x^k$$

where $C(n,k)$ is the number of permutations of $[n]$ with k cycles, and we skip term $k = 0$ because $C(n,0) = 0$.

Now lets take the derivative of both sides,

$$\sum_{i=0}^{n-1} x(x+1)\cdots\widehat{(x+i)}\cdots(x+n-1) = \sum_{k=1}^{n} kC(n,k)x^{k-1}$$

where $\widehat{(x+i)}$ means we skip that term.

If we plug the value $x = -1$, the RHS is closely related to what we want, because when n is even, the parity of the number of cycles of a permutation is the same as the parity of the permutation itself, and if n is odd, is the opposite.

When we plug -1 in the polynomial almost all terms of the LHS are cero, except the one we skip the factor $(x+1)$, so we get

$$-(n-2)! = \sum_{k=1}^{n} kC(n,k)(-1)^{k-1}$$

$$= \sum_{\text{k, odd}} kC(n,k) - \sum_{\text{k, even}} kC(n,k)$$

$kC(n,k)$ is k times the number of permutations with k cycles, so it counts the total number of cycles of the permutations with k cycles, so because of what we said earlier,

$$\text{if } n \text{ is even } \sum_{\text{k, odd}} kC(n,k) = o_n \text{ and } \sum_{\text{k, even}} kC(n,k) = e_n,$$

then $-(n-2)! = o_n - e_n \Rightarrow e_n - o_n = (n-2)!$

$$\text{and if } n \text{ is odd } \sum_{\text{k, odd}} kC(n,k) = e_n \text{ and } \sum_{\text{k, even}} kC(n,k) = o_n,$$

then $-(n-2)! = e_n - o_n$, and in both cases we get what we wanted.

(b) Give a bijective proof of (a).

We know $f : S_n \longrightarrow S_n : w \longmapsto w(12)$ the composition with the transposition (12) si a bijection between even and odd permutations, but lets see more more detailed other bijections defined by this mapping. A permutation having 1 and 2 in the same cycle is mapped in one having them in different cycles, and the converse is true, too.

Lets say ed_n = number of even permutations with 1 and 2 in different cycles, es_n= number of even permutations with 1 and 2 in the same cycle, and od_n, os_n defined similarly. let Ced_n= total number of cycles of even permutations with 1 and 2 in different cycles , and the others be defined similarly.

With the mapping f we have bijections that shows the following equalities

$$ed_n + es_n = od_n + os_n$$

$$ed_n = os_n$$

$$es_n = od_n$$

And lets change the problem in terms of the new definitions.

$e_n = Ced_n + Ces_n$ and $o_n = Cod_n + Cos_n$

If we have a permutation with 1 and 2 in different cycles and we map it through f the cycles of 1 and 2 form a new cycle, so the number of cycles is decreased by one. Then

$$Ced_n = Cos_n + ed_n$$

similarly

$$Ces_n = Cod_n + es_n$$

So we obtain $e_n - o_n = ed_n - es_n$ meaning that what we want is equal to the number of even permutations with 1 and 2 in different cycles minus the number of even permutations with 1 and 2 in the same cycle. Now lets prove $ed_n - es_n = (-1)^n(n-2)!$ by induction.

If $n = 2$ there is only one even permutation, $(1)(2)$ and have 1 and 2 in different cycles, so $ed_2 = 1$ and $es_2 = 0$, so it is true for $n = 2$. Now suppose $ed_{n-1} - es_{n-1} = (-1)^{n-1}(n-3)!$ We have the following recurrence relation, $ed_n = ed_{n-1} + od_{n-1}(n-1)$ depending if n is the only one in its cycle or not, if it is, then the parity of the permutation doesn't change and we have ed_{n-1} possibilities for the other $n-1$, and if n is not the only one in its cycle, then when we add it to another cycle the parity of the permutation changes, so we have od_{n-1} possibilities for the $n-1$ when we erase n and to put it again we have $n-1$ possibilities for choosing the image of n.

$$ed_n = ed_{n-1} + od_{n-1}(n-1) = ed_{n-1} + es_{n-1}(n-1)$$

the last change because of the equalities we had at the beginning. Similarly we obtain

$$es_n = es_{n-1} + os_{n-1}(n-1) = es_{n-1} + ed_{n-1}(n-1)$$

With this we get
$ed_n - es_n = ed_{n-1} + es_{n-1}(n-1) - es_{n-1} - ed_{n-1}(n-1)$
$= (ed_{n-1} - es_{n-1})(1-(n-1)) = (-1)^{n-1}(n-3)!(2-n) = (-1)^n(n-2)!$