2. (Counting binary words by runs) Collaborators: Nina Cerutti, Hannah Winkler
A *binary word* is a word consisting of 0s and 1s. A run is a maximal string of consecutive 1s. For example the word 11010111011 has 4 runs. Find the number of binary words having exactly m 0s, n 1s, and k runs.

Proof. This is a fun problem and as usual it is helpful to think of an example. First allow me to define a sequence of zeros to be a *stop*. So for instance the following binary word would contain 3 stops.

$$000111100110 \text{ A binary word with 3 stops.}$$

Suppose we wanted to generate a binary word with $n = 4$ 1's and m 0's and k runs. Now I want to consider all of the possible sequences of runs. In other words I want to consider all the ways of grouping the 4 1's. This results in the following possibilities:

Grouping of 4 1's	Number of Runs
$1-1-1-1$	4 runs
$11-1-1$	3 runs
$1-11-1$	3 runs
$1-1-11$	3 runs
$11-11$	2 runs
$111-1$	2 runs
$1-111$	2 runs
1111	1 run

But notice that the different groupings of 4 1's is just the number of compositions of 4. That is there are 2^{4-1} ways of grouping the 1's. And a consequence of this is that the number of binary words with n 1's and k runs is equal to k compositions of n or $\binom{n-1}{k-1}$. Also note that for k runs there are a minimum of $k-1$ and a maximum of $k+1$ stops.

Grouping of 4 1's	Number of Runs and Stops	k compositions of 4
$1-1-1-1$	4 runs, $3 \leq$ stops ≤ 5	$1+1+1+1$
$11-1-1$	3 runs, $2 \leq$ stop ≤ 4	$2+1+1$
$1-11-1$	3 runs, $2 \leq$ stops ≤ 4	$1+2+1$
$1-1-11$	3 runs, $2 \leq$ stops ≤ 4	$1+1+2$
$11-11$	2 runs, $1 \leq$ stops ≤ 3	$2+2$
$111-1$	2 runs, $1 \leq$ stops ≤ 3	$3+1$
$1-111$	2 runs, $1 \leq$ stops ≤ 3	$1+3$
1111	1 run, $0 \leq$ stops ≤ 2	4

Now we want to take into account the placement of the m 0's. In order for a binary word to contain k runs it must have at least $k-1$ stops with one stop placed in between each run. This means that $k-1$ of the m zeroes must be placed in between each of the k runs. (Note that if $k-1 > m$ then there are zero binary words.) For the remaining $(m-(k-1))$ 0's, if there are any, they can be placed on either side of each of the runs, that is there are $k+1$ places to distribute the remaining 0's.

So for a binary word consisting of n 1's, 3 runs and m 0's we have

$$\underbrace{0\cdots0}_{m_1}1\cdots1\underbrace{0\cdots0}_{m_2}1\cdots1\underbrace{0\cdots0}_{m_3}1\cdots1\underbrace{0\cdots0}_{m_4}$$

where $m_2, m_3 \geq 1$ and $m_1+m_2+m_3+m_4 = m$. For this case there are place places where we can distribute the $m-2$ 0's. Suppose $m = 5$ the possible distributions of $(5-2)$ 0's would be the following:

$1+1+1+0$	$0+2+0+1$	$3+0+0+0$
$0+1+1+1$	$1+0+2+0$	$0+3+0+0$
$2+1+0+0$	$0+1+2+0$	$0+0+3+0$
$2+0+1+0$	$0+0+2+1$	$0+0+0+3$
$2+0+0+1$	$1+0+0+2$	$1+0+1+1$
$1+2+0+0$	$0+1+0+2$	$1+1+0+1$
$0+2+1+0$	$0+0+1+2$	

But these are just the weak composition of 3 into 4 parts. In general this is a $m-(k-1)$ weak composition into $k+1$ parts or a weak $k+1$ composition of $m-(k-1)$.

So the number of binary words consisting of n 1's with k runs and m 0's is equal to

$$\boxed{\binom{n-1}{k-1}\binom{m-(k-1)+(k+1)-1}{k+1-1} = \binom{n-1}{k-1}\binom{m+1}{k}}$$

□