1a Give a combinatorial proof that for any positive integers $n \geq k$,

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

Solution. A car lot has $n > 0$ different black cars. How many different ways are there to put $k \leq n$ of the cars into a garage so that one of the k cars in the garage is also painted red?

We could choose k cars out of n to move to the garage in $\binom{n}{k}$ ways. Then there are k choices for which to paint one of them red. Thus we can accomplish the task in $k\binom{n}{k}$ ways.

Instead we could first choose which car to paint red and move into the garage; this can be done in n ways. Then there are $n-1$ cars left and we need to choose $k-1$ of them to move into the garage with the red car. So we can accomplish the task in $n\binom{n-1}{k-1}$ ways.

Since we were counting the same thing,

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Dividing by k,

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}.$$

1b Give a combinatorial proof that for any positive integers $n \geq k$,

$$\sum_{l=k}^{n}\binom{l}{k} = \binom{n+1}{k+1}$$

Solution. Let's count the number of $(k+1)$-element subsets of $[n+1] = \{1, 2, ..., n+1\}$.

This can be counted straightforwardly as $\binom{n+1}{k+1}$.

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On the other hand, we can count the total number of subsets with the desired property by counting how many we have with a certain largest element and adding them together. Each of the $(k+1)$-element subsets of $[n+1] = \{1, 2, ..., n+1\}$ will have a largest element between $k+1$ and $n+1$ (inclusive). Once we determine what the largest element l of the set is, we must choose k additional elements which are less than the largest element to make the $(k+1)$-element subset. There are $l-1$ elements smaller than l to choose from. Thus the number of $(k+1)$-element subsets of $[n+1] = \{1, 2, ..., n+1\}$ is $\sum_{l=k+1}^{n+1} \binom{l-1}{k} = \sum_{l=k}^{n} \binom{l}{k}$.

Since we're counting the same objects, $\sum_{l=k}^{n} \binom{l}{k} = \binom{n+1}{k+1}$.