3. **(A generating function identity) Let k be a fixed positive integer. Prove the identity**

$$\sum_{n_1,\dots,n_k\geq 0} min(n_1,...,n_k)x_1^{n_1}\cdots x_k^{n_k} = \frac{x_1x_2\cdots x_k}{(1-x_1)(1-x_2)\cdots(1-x_k)(1-x_1x_2\cdots x_k)}$$

where we are summing over all k-tuples of non-negative integers $n_1,...,n_k$, and $min(n_1,...,n_k)$ denotes the smallest number among $n_1,...,n_k$.

Before we begin our proof let's quickly note that if at least one $n_i = 0$, then $min(n_1,...,n_k) = 0$.
Let

$$A(x_1,...,x_k) = \sum_{n_1,\dots,n_k\geq 0} min(n_1,...,n_k)x_1^{n_1}\cdots x_k^{n_k}$$

Then we have that $(1-x_1x_2\cdots x_k)A(x_1,...,x_k) =$

$$A(x_1,...,x_k) - x_1x_2\cdots x_kA(x_1,...,x_k)$$
$$\sum_{n_1,\dots,n_k\geq 0} min(n_1,...,n_k)x_1^{n_1}\cdots x_k^{n_k} - \sum_{n_1,\dots,n_k\geq 0} min(n_1,...,n_k)x_1^{n_1+1}\cdots x_k^{n_k+1}$$
$$\sum_{n_1,\dots,n_k\geq 1}\Big[min(n_1,...,n_k) - min(n_1-1,...,n_k-1)\Big]x_1^{n_1}\cdots x_k^{n_k}$$

Now if we have that $min(a_1,...,a_k) = a_i$ for some i, then we must have that $a_i - 1 \leq a_j - 1$ for all j. Thus $min(a_1-1,...,a_k-1) = a_i-1$. From this we have that $min(n_1,...,n_k)-min(n_1-1,...,n_k-1) = 1$. Therefore we have that $(1-x_1x_2\cdots x_k)A(x_1,...,x_k) =$

$$\sum_{n_1,\dots,n_k\geq 1} x_1^{n_1}\cdots x_k^{n_k}$$
$$\left(\sum_{n\geq 1} x_1^n\right)\left(\sum_{n\geq 1} x_2^n\right)\cdots\left(\sum_{n\geq 1} x_k^n\right)$$
$$\left(\tfrac{x_1}{1-x_1}\right)\left(\tfrac{x_2}{1-x_2}\right)\cdots\left(\tfrac{x_k}{1-x_k}\right)$$
$$\tfrac{x_1x_2\cdots x_k}{(1-x_1)(1-x_2)\cdots(1-x_k)}$$

Hence we have that $A(x_1,...,x_k) = \frac{x_1x_2\cdots x_k}{(1-x_1)(1-x_2)\cdots(1-x_k)(1-x_1x_2\cdots x_k)}$.