2a. *Worked with Emily* Let w_n be the number of words of length n in the alphabet $\{A, B, C\}$ which do not have two consecutive consonants.

If we focus on what happens at the beginning of a word of length n, it can begin with an A, and then there is any word of length $n-1$ following it. There are no restrictions of what can follow the A because it's a vowel. The word can also begin with a B, but then it is mandatory that the next letter is an A, since we cannot have two consecutive consonants. So we have BA followed by a word of length $n-2$, with no restrictions of what follows the A. Similarly, the word can begin with CA, and be followed by any word of length $n-2$.

Therefore, w_n, the total number of words of length n, can be given recursively by

$$w_n = w_{n-1} + 2w_{n-2}, \qquad (\text{for } n \geq 2).$$

2b. *Worked with Emily* The generating function for w_n is

$$W(x) = w_0 + w_1 x + w_2 x^2 + w_3 x^3 + ...$$

Using our recursion from part (a), we have

$$\begin{aligned} W(x) = 1 + 3x + w_1 x^2 + w_2 x^3 + ... \\ +2w_0 x^2 + 2w_1 x^3 + ... \end{aligned}$$

The first line can be rewritten as $1 + 2x + xW(x)$ and the second line can be rewritten as $2x^2 W(x)$. So,

$$\begin{aligned} W(x) &= 1 + 2x + xW(x) + 2x^2 W(x) \\ \implies W(x) &= \frac{1+2x}{1-x-2x^2}. \end{aligned}$$

2c. *Worked with Emily* Using partial fractions, we want to rewrite $W(x)$ as a geometric series, so that we will be able to explicitly see the formula for w_n as the coefficient of x^n in the series.

$$W(x) = \frac{1+2x}{1-x-2x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

$$= \frac{A(1-\beta x) + B(1-\alpha x)}{1-(\alpha+\beta)x-(-\alpha\beta)x^2}$$

for some A, B, α, β. From the denominator, we get the system of equations $\alpha + \beta = 1$ and $-\alpha\beta = 2$. Using substitution, we get that $\alpha = 2$ and $\beta = -1$.

We can rewrite the numerator as $(A+B)-(A\beta+B\alpha)x$, and so we get the system of equations $A + B = 1$ and $A\beta + B\alpha = -2$. Again, through substitution and using our results for α, β, we get $A = \frac{4}{3}$ and $B = \frac{-1}{3}$. So,

$$W(x) = \frac{1+2x}{1-x-2x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

$$\implies W(x) = A\sum_{n\geq 0} \alpha^n x^n + B\sum_{n\geq 0} \beta^n x^n = \sum_{n\geq 0} (A\alpha^n + B\beta^n)x^n$$

$$\implies w_n = A\alpha^n + B\beta^n$$

$$\implies w_n = \frac{4}{3}2^n - \frac{1}{3}(-1)^n$$

$$\implies w_n = \frac{1}{3}\left(2^{n+2} + (-1)^{n+1}\right)$$