
coxeter groups . 2008

final projects

. san francisco state university .
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For the final project of the course, you will develop a solid understanding of a particular aspect of Coxeter group theory that interests you. You may, for instance:

- Understand the background and significance of an open problem in Coxeter groups, and solve it, or achieve some partial progress.
- Understand the current state of the art in a branch of Coxeter groups, and present it in a clear, concise, and useful survey.
- Find a new way of thinking about or proving a known result.
- Write a computer program that will be useful to researchers in Coxeter groups.

due dates.

- o Wednesday, April 9: rough outline. (1-2 pages)
- o Friday, May 16: final project. (~10 pages in LaTeX, 11pt, single space)

some suggested sources.

The existing literature on Coxeter groups is gigantic, and there are many topics which are within your reach. Many papers and books contain interesting open problems which you can understand and think about. Search on Google Scholar, the mathematics arXiv, and the American Mathematical Society's mathscinet. Pointers to several other useful books and online resources are on the course website. A careful and detailed study of a difficult problem in the book or homework is another possible starting point.

In my suggestions below I have posted some pointers to the literature which you should easily find on google, the arxiv, or mathscinet. Let me know if you have looked carefully and still can't find some of the references I mention.

some suggested topics.

Here are some possible topics for a final project. Feel free to choose your own topic, but you'll need me to approve it before you start working on it. As you can see from the list, I am flexible about the topics that you choose - but you must prove to me that you learned a lot of mathematics by working on the project! In your proposal you will tell me what you plan to do and I will give you concrete feedback on your specific plan.

1. Understand the Tits cone of an infinite Coxeter group W , including detailed diagrams (and proofs!) for some interesting examples of W .
2. Study the problem of writing a Coxeter group element as a product of (not necessarily simple) reflections. Study the “reflection length function”. See the papers of Dyer and Howlett.
3. Study the problem of writing a Coxeter group element as a product of sets of commuting simple reflections. Study the “automatic length function” and its reciprocity. See Glover-Scott.
4. Study hyperbolic reflection groups.
5. Study complex reflection groups.
6. Study the invariant ring of a Coxeter group. See Kane’s book.
7. Study D. Armstrong’s notion of convexity in terms of the sorting order on a Coxeter group.
8. Study Björner’s work on greedoids and Coxeter groups.
9. Study the topology of Knutson and Miller’s “subword simplicial complex”. (Requires some basic algebraic topology which you can probably teach yourself.)
10. Geometric group theory studies Coxeter groups by making them act nicely on geometric spaces, and studying the geometry of those spaces. See Davis’s new book for *many* possible topics. For example, understand the following (roughly correctly stated) theorem: The word problem for a Coxeter group W can be solved in linear time if and only if W acts on a space of negative curvature.
11. Study the matroid of a root system, compute various invariants such as the Tutte polynomial. See Ardila’s thesis and de Concini and Procesi’s recent article on this.
12. Study an aspect of Coxeter matroids, a generalization of matroid theory to Coxeter groups. There are *many* open problems in this area, which you can find in the book by Borovik, Gelfand, and White.
13. Study the various enumerative problems in the combinatorics of the W -Catalan numbers. See for example the notes of Fomin and Reading, and work of Athanasiadis and of Reading.
14. Understand the basics of Fomin and Zelevinsky’s theory of cluster algebras and the role played by Coxeter groups.
15. Study Fomin and Zelevinsky’s generalized associahedra of type W , and the realization of Hohlweg, Lange, and Thomas.

16. Study the graph associahedron of type W and its role in tropical geometry. See the paper by Ardila, Reiner, Williams and the paper by Feichtner and Sturmfels.
17. Understand the role of Coxeter groups in the theory of Lie algebras and/or Lie groups. (Requires some knowledge of Lie theory.)
18. Study in detail the combinatorics of the hyperoctahedral group S_n^B .
19. Study Bidigare, Hanlon and Rockmore's random walks on a hyperplane arrangement and its relationship to card shuffling.
20. Study the representation theory of Coxeter groups. A possible starting point is the recent paper of Adin, Brenti, and Roichman.
21. Study the topology of intervals in the Bruhat order. See Chapter 2 of the book, Fomin and Shapiro's paper, and the very recent paper of Hersh.
22. Investigate the order dimension of the Bruhat orders and compute them or obtain bounds in the open cases. See Exercise 2.24 in Björner-Brenti.
23. Study the open problems listed as Exercises 2.37, 3.6, 3.8, 3.11b, 4.23b in Björner-Brenti.
24. Understand the presentation of Coxeter group elements in their normal forms. Start with Chapter 3 in Björner-Brenti.
25. Study the topology and combinatorics of Schubert varieties. Understand how it is used to solve enumerative problems in geometry. (Requires some background in these fields.)
26. Study Kazhdan-Lusztig polynomials. (See Chapter 5 of Björner-Brenti.)
27. Study the refined length generating functions for Coxeter groups. (See Chapter 6 of Björner-Brenti.)
28. Study the number of reduced decompositions of the elements of S_n . (See Chapter 5 of Björner-Brenti.)
29. Carry out a careful study of the exceptional Coxeter groups.
30. Study the Hecke algebra of a Coxeter group. (I can tell you a concrete open problem about this.)
31. Study Billera and Brenti's quasisymmetric function of a Coxeter group and the connection with Hopf algebras.
32. Study the alternating subgroup of a Coxeter group. You might start with Brenti, Reiner, and Roichman's paper.

33. Revisit Coxeter's work on regular polytopes.
34. Study Bruhat-Tits buildings.
35. Study Bachmann's project to establish the foundations of geometry in terms of reflections. SFSU professor J. Smith can provide references on this topic.
36. Study Thomsen's equation, an equation of length 22 satisfied by the reflections a, b, c of across the edges of an arbitrary triangle. See J. Smith's paper, where he shows how to derive all such equations and proves this is the shortest one.
37. An (as far as I know) open problem in plane geometry that may be related. Given a triangle T in the plane. Can we bring an arbitrary point of the plane into T by successively reflecting it across the three lines containing the sides of T ?
38. Study the classification of frieze and wallpaper groups. See the books by J. Smith and by Grunbaum and Shephard.
39. What is a permutahedron doing in the playground of 23rd St. and Folsom St. in San Francisco? (And many other playgrounds.) What is the bigger structure that contains it? Research the history and the mathematics behind this.
40. What are Coxeter groups doing in the San Francisco Museum of Modern Art? Explore the mathematical aspects of Olafur Eliasson's art related to Coxeter groups.
41. Study Borovik's writings on the cognitive aspects of Coxeter group theory.