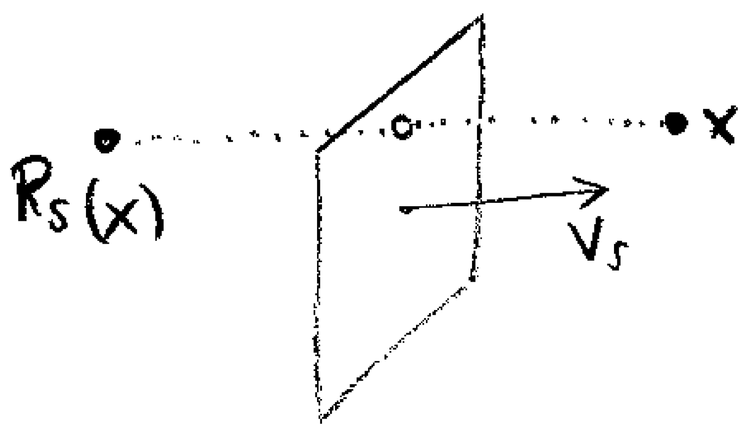
Reflection groups, Coxeter groups (Parenthesis)



hyperplane H_s: $v_s \cdot x = 0$

reflection across H_s:

$$R_s(x) = x - 2\frac{v_s \cdot x}{v_s \cdot v_s} v_s$$

Take $H_1, \dots, H_r$ hyperplanes

$R_1, \dots, R_r$ reflections across them.

Exercise: $R_i R_j$ = rotation by α_{ij} (angle of v_i, v_j)

so $(R_i R_j)^{m_{ij}} = e$ $\quad m_{ij} = \begin{cases} \min m \text{ s.t. } m\alpha_{ij} = 2k\pi \\ \infty \quad \text{if no such } m \end{cases}$

Reflection Rep'n of Coxeter groups

<u>Goal</u>: To think of any Coxeter group geometrically.

(But we need to allow more general reflections)

Given (W, S)

If $S = \{s_1, \dots, s_r\}$ $\quad m(s_i, s_j) = m_{ij}$

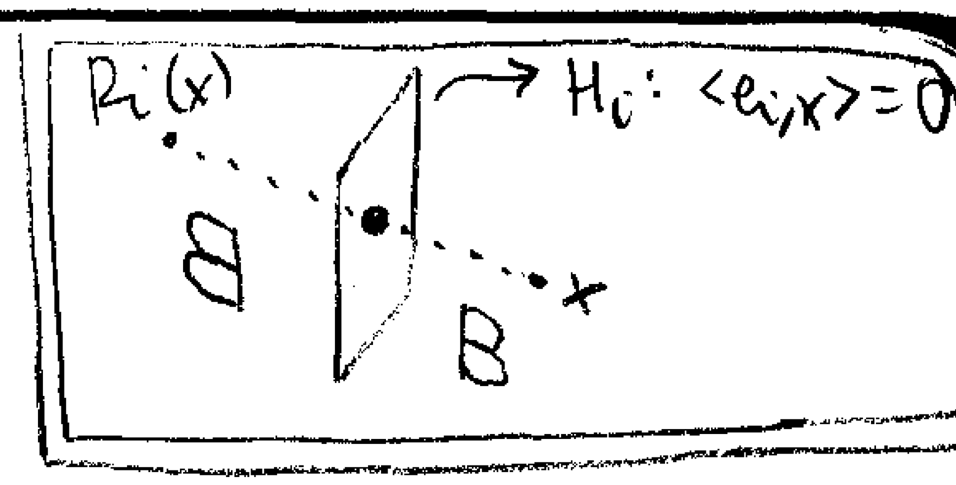
Let V = vector space with basis $e_1, \dots, e_r$

Inner product: $\langle e_i, e_j \rangle = -\cos\left(\frac{\pi}{m_{ij}}\right)$

$-1 \quad$ if $m_{ij} =$

$\langle e_i, e_i \rangle = 1$

Define the "reflections"



$$R_i(x) = x - 2\langle e_i, x\rangle e_i$$

$$R_i \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ -2\cos\left(\frac{\pi}{m_{1i}}\right) & -2\cos\left(\frac{\pi}{m_{2i}}\right) & \cdots & -1 & \cdots & -2\cos\left(\frac{\pi}{m_{ni}}\right) \\ & 0 & & 1 & & 0 \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$R_i R_j \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{array}{c c} & \begin{array}{cccccc} & & i & j & & \end{array} \\ \begin{array}{c} \\ \\ i \\ j \\ \\ \\ \end{array} & \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & -1+4c^2 & 2c & & \\ & & -2c & -1 & & \\ 0 & & & & 1 & \\ & & & & & 1 \end{bmatrix} \end{array} \quad c = \cos\left(\frac{\pi}{m_{ij}}\right)$$

Exercise: eigenvalues are $\cos\frac{2\pi}{m_{ij}} \pm i \sin\frac{2\pi}{m_{ij}} = e^{\pm\frac{2\pi i}{m_{ij}}}$

($R_i R_j$ is a rotation by $\frac{2\pi}{m_{ij}}$)

So $(R_i R_j)^{m_{ij}} = I$

By universality, get

$$\phi: W \to GL(V)$$
$$s_i \mapsto R_i$$

Thm

Given a Coxeter system (W,S) with matrix m

a) The order of $s_i s_j$ is $m(s_i, s_j)$

b) $s_i \neq s_j$ for $i \neq j$ (S minl set of gens)

Pf.

a) If $(s_i s_j)^k = e$ in W $1 \leq k < m$)

then $(R_i R_j)^k = I$ in $GL(V)$ $(k < m)$ $\Rightarrow\Leftarrow$

rotation by $0 < \frac{2k\pi}{m} < 2\pi$

(For $m(i,j) = \infty$ a bit trickier.

$R_i R_j$ has $\lambda s = 1, 1$

$R_i R_j = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 3 & 2 \\ & & -2 & -1 \\ & & & & 1 \end{bmatrix}$ not diagonalizable

so $(R_i R_j)^k$ is not diagonalizable

$\Rightarrow$ not the identity.)

b) Sup $s_i = s_j$

$\Rightarrow (s_i s_j)^2 = e$

$\Rightarrow m(i,j) = 1$

$\Rightarrow i = j.$