(1) $s_1 \cdots s_k$ not reduced

let $s_i \cdots s_k$ be max not reduced

$$\ell(s_i \cdots s_k) < \ell(s_{i+1} \cdots s_k)$$

exchange $\Rightarrow$ $s_i \cdots s_k = s_{i+1} \cdots \hat{s}_j \cdots s_k$

$$s_1' \cdots s_k' = s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$$

① = ③, ② = ③ are shorter so are ok ✓

(2) $s_1 \cdots s_k$ reduced

Can assume $s_1' \neq s_1$, so

$$s_1' s_1 \cdots s_k = s_1 \cdots \hat{s}_i \cdots s_k$$

$$s_1' \cdots s_k' = s_1 \cdots s_k = s_1' s_1 \cdots \hat{s}_i \cdots s_k$$

① = ③ shorter

② = ③ shorter unless $i = k$

So we are ok unless ② = ③ is:

$$s_1' s_1 \cdots s_{k-1} = s_1 \cdots s_k \quad \text{Is it ok?}$$

Repeat.

$s_1 \cdots s_k = s_1' \cdots s_k'$ ok?

ok, or $\rightarrow$ $s_1' s_1 \cdots s_{k-1} = s_1 \cdots s_k$ ok?

ok, or $\rightarrow$ $s_1 s_1' s_1 \cdots s_{k-2} = s_1' s_1 \cdots s_{k-1}$ ok?

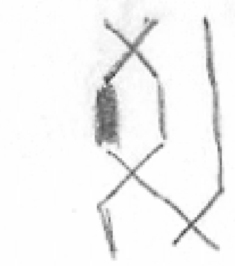
$\rightarrow$ ⋮

$\rightarrow$ $s_1 s_1' s_1 s_1' \cdots = s_1' s_1 s_1' s_1 \cdots$ ok?

Yes!

Ex S_n^B is a Coxeter group.

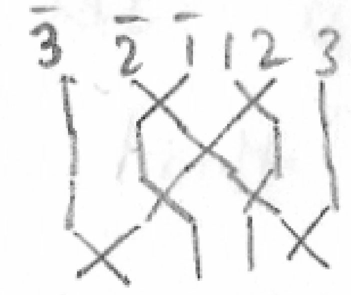
Pf: represent elements by symmetric braids



$s_2 s_1 s_0 s_1$

$1\,3\,\bar{2}$

$\longleftrightarrow$



$s_2 s_1 s_0 s_1$

$2\,\bar{3}\,\bar{1}\,1\,3\,\bar{2}$

$12 \quad \bar{1}\bar{2}$

$\bar{1}1$

$\bar{1}\bar{2} \quad \bar{1}2$

$\bar{2}3 \quad 2\bar{3}$

length = # of "inversions" wrt $\bar{n} < \cdots < \bar{2} < \bar{1} < 1 < 2 < \cdots < n$

- $(i\,\bar{i})$ counts as one
- $\{(i\,j), (-i\,-j)\}$ counts as one

Ex: $\ell(1\,3\,\bar{2})$: $\quad 2\,\bar{3}\,\bar{1}\,1\,3\,\bar{2}$ $\qquad \bar{3} < \bar{2} < \bar{1} < 1 < 2 < 3$

$1 \rightarrow$ $\quad 3 \rightarrow$

$\ell(1\,3\,\bar{2}) = 4$

If $\ell(tw) < \ell(w)$, it's because t undoes a crossing that already took place

$$w = s_1 \cdots s_k$$
$$\downarrow$$
$$tw = s_1 \cdots \hat{s}_i \cdots s_k$$

$\Rightarrow$ Exchange property holds

$\Rightarrow$ Coxeter group