Nice corollaries:

(i) Any $w = s_1 \cdots s_k$ contains a reduced expression as a subword

(ii) If $w = s_1 \cdots s_k = s_1' \cdots s_k'$ are reduced then $\{s_1, \ldots, s_k\} = \{s_1', \ldots, s_k'\}$

(iii) No elt of S is generated by the others.

(i) clear

(ii) Suppose $s_i \notin \{s_1', \ldots, s_k'\}$ i min.

Recall: the t's that shorten w are:

$\{s_1 \cdots s_i \cdots s_1 \quad (1 \le i \le k)\}$

$\{s_1' \cdots s_j' \cdots s_1' \quad (1 \le j \le k)\}$

$\Rightarrow \quad s_1 \cdots s_i \cdots s_1 = s_1' \cdots s_j' \cdots s_1'$

$s_i =$ word in $\{s_1' \cdots s_k'\}$

some subword $s_t' = s_i$ ✓

(iii) clear from (ii)

Theorem

W group

S set of generators of order 2

TFAE: (i) (W,S) is a Coxeter system

(ii) (W,S) has the exchange property.

(iii) (W,S) has the deletion property.

(i) $\Rightarrow$ (ii) done

(ii) $\Rightarrow$ (iii) done above

(iii) $\Rightarrow$ (ii) $\ell(s s_1 \cdots s_k) < \ell(s_1 \cdots s_k) = k$

$$\Rightarrow s s_1 \cdots s_k = \begin{cases} s s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k & \text{(no)} \\ s_1 \cdots \hat{s}_j \cdots s_k & ✓ \end{cases}$$

(ii) $\Rightarrow$ (i) Sup exchange, deletion.

Let $m(s,s') =$ order of ss' if finite

Need: Every rel in W follows from $(ss')^{m(s,s')} = e$

Take a relation. $\underbrace{\qquad}_{\text{even by deletion}} = e$. Rewrite

$$s_1 s_2 \cdots s_k = s_1' \cdots s_k'$$

Induct on k. ($k=1$ trivial)