(W,S) Coxeter system

T reflections $\leftrightarrow$ palindromic words

$\pi : W \to S_T^B$

$w \mapsto \pi_w \qquad \pi_w(t) = \pm w t w^{-1}$

(in book, $\eta_w(t)$)

$\operatorname{sgn}(w,t) = (-1)^{(\#\text{ of times } t = s_k \cdots s_i \cdots s_k)}$

for any $w = s_1 \cdots s_k$

Next goal:

Use this signed permut rep'n to understand the different words for an element $w \in W$

Parity. ∘ All words for w have the same length parity. Call w even or odd.

∘ Even words form the "alternating subgroup"

Length The length $\ell(w)$ is the min k for which we can write

$$w = s_1 s_2 \cdots s_k$$

A word of min length is a reduced word

Properties:

∘ $\ell(sw) = \ell(w) \pm 1$ for $s \in S$

∘ $\ell(uv) \le \ell(u) + \ell(v)$

∘ $\ell(w^{-1}) = \ell(w)$

Read off W info from $\pi : W \to S_T^B$

Theorem $\ell(w) = |\{t \in T \mid \ell(tw) < \ell(w)\}|$

\# of reflections that shorten w

$= |\{t \in T \mid \operatorname{sgn}(w^{-1}, t) = -1\}|$

Lemma: $w \in W,\ t \in T$

$\ell(tw) < \ell(w) \iff \operatorname{sgn}(w^{-1}, t) = -1$

Pf.

$\Leftarrow$ Let $w = s_1 \cdots s_d$ reduced, $w^{-1} = s_d \cdots s_1$

t appears an odd number of times as

$t = s_1 \cdots s_i \cdots s_1$

$\Rightarrow tw = s_1 \cdots \cancel{s_i \cdots s_1 s_1 \cdots s_i} \cdots s_d$

$= s_1 \cdots s_{i-1} s_{i+1} \cdots s_d$ shorter ✓

$\Rightarrow \ell(tw) < \ell(w) = \ell(ttw) \Rightarrow \operatorname{sgn}((tw)^{-1}, t) = 1$

$\to \pi_{(tw)^{-1}}(t) = (tw)^{-1} t tw$

$\pi_{w^{-1}} \pi_t(t) = w^{-1} t w$

$\pi_{w^{-1}}(-t) = w^{-1} t w \to \operatorname{sgn}(w^{-1}, t) = -1$ ∎

Lemma 2

If $w = s_1 s_2 \cdots s_k$ reduced (k min) and $t \in T$,

These are equivalent:

(a) $\ell(tw) < \ell(w)$

(b) $tw = s_1 \cdots \widehat{s_i} \cdots s_k$, some i

(c) $t = s_1 \cdots s_i \cdots s_1$, some i

Pf

(a) $\Rightarrow \ell(tw) < \ell(w) \Rightarrow \operatorname{sgn}(w,t) = -1$

$\Rightarrow t = s_1 \cdots s_i \cdots s_1$

$\Rightarrow tw = s_1 \cdots \widehat{s_i} \cdots s_k \Rightarrow$ (b)

(b) $\Rightarrow$ (a) obvious

(b) $\Leftrightarrow$ (c): Compute. ∎

For theorem, it remains to show

$$s_1 \cdots s_i \cdots s_1 \neq s_1 \cdots s_i s_{i+1} \cdots s_j \cdots s_{i+1} s_i \cdots s_1$$

$$e \neq s_i \cdots s_j \cdots s_{i+1}$$

$$s_{i+1} \cdots s_{j-1} \neq s_i \cdots s_j \qquad \text{ok by reducedness!}$$

∎

Two important properties

① Exchange property:

If $w = s_1 \cdots s_k$, $s \in S$ and $\ell(sw) < \ell(w)$ then $sw = s_1 \cdots \widehat{s_i} \cdots s_k$ for some i.

①' Strong exchange property:

Same with $t \in T$ instead of $s \in S$

② Deletion property:

If $w = s_1 \cdots s_k$ and $\ell(w) < k$ then $w = s_1 \cdots \widehat{s_i} \cdots \widehat{s_j} \cdots s_k$ for some $i \neq j$.

①' follows from proof of Lemma 2

①' $\Rightarrow$ ① ✓

①' $\Rightarrow$ ②: Take $s_i \cdots s_k$ not reduced, i max

$\Rightarrow \ell(s_i \cdots s_k) < \ell(s_{i+1} \cdots s_k)$

$\Rightarrow s_i \cdots s_k = s_{i+1} \cdots \widehat{s_j} \cdots s_k$ ✓