Pf Induct. $1\times 1 \to$ clear

Sup true for $(n-1)\times(n-1)$. Let $A = \begin{bmatrix} A' & \\ & \end{bmatrix}$

$\Downarrow$: A' is pos. def because

$$\begin{bmatrix} x' \end{bmatrix} \begin{bmatrix} A' \end{bmatrix} \begin{bmatrix} x' \end{bmatrix} = \begin{bmatrix} x' & 0 \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x' \\ 0 \end{bmatrix} > 0.$$

So by induction, first $n-1$ ppal minors > 0

last one is $\det A = \lambda_1 \cdots \lambda_n > 0$. ∎

$\Uparrow$: The $(n-1)$-dim subspace $U = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \right\}$

has $x^T A x = x'^T A' x' > 0$.

Let an orthonormal basis of eigenvecs for A be

$v_1, \ldots, v_k, v_{k+1}, \ldots, v_n$

$\lambda_1, \ldots, \lambda_k \leq 0 \quad \lambda_{k+1}, \ldots, \lambda_n > 0$

Then $\operatorname{span}(v_1, \ldots, v_k)$ has $x^T A x \leq 0$

Since these two don't intersect, $k \leq 1$

so A has at most one $\lambda \leq 0$.

But $\det A = \lambda_1 \cdots \lambda_n > 0$ so I can't have any $\lambda \leq 0$. ∎

By Sylvester's criterion, it is time to compute some determinants!

Given a Coxeter graph $\overset{m_{ij}}{i \text{---} j}$, when is the matrix $A = \left[-\cos\left(\frac{\pi}{m_{ij}}\right) \right]_{1 \leq i,j \leq n}$ positive definite?

Call these "pos def" graphs, they correspond to the finite Coxeter groups.

Theorem The irreducible finite Coxeter groups are:

A_n

B_n

D_n

E_6

E_7

E_8

F_4

H_3

H_4

$I_2(m)$