Lemma 3. If $Z_G(U)=\{cI : c\neq 0\}$ then there is a unique non-degenerate G-invariant bilinear form up to scaling.

→ no $u\in V$ such that $\langle u,v\rangle=0$ for all $v\in V$. $(V^\perp=0)$

Pf. Existence: Lemma 1

Uniqueness: Sup. $(\,,\,)$, $\langle\,,\,\rangle$ do.

nondeg Let $u_1,..,u_n$ be a basis for V.

Let $v_1,\to v_n$ be dual basis wrt $(\,,\,)$

$w_1,\to w_n$ $\langle\,,\,\rangle$

so $(u_i,v_i)=1$ all i $\langle u_i,w_i\rangle=1$

$(u_i,v_j)=0$ all $i\neq j$ $\langle u_i,w_j\rangle=0$.

Def $\varphi: U\to U$ by $\varphi(v_i)=w_i$. Then

$$(u,v)=\langle u,\varphi(v)\rangle \qquad u,v\in V$$

Now

$$\langle u, g\varphi(v)\rangle = \langle g^{-1}u,\varphi(v)\rangle$$
$$= (g^{-1}u, v)$$
$$= (u, gv) = \langle u,\varphi(gv)\rangle$$

for all u, so

$$g\varphi v = \varphi g v$$

$\to \varphi\in Z_G(U) \to \varphi(v)=cv$

$$(u,v)= c\langle u,v\rangle$$

☒

Lemma 4. If (W,S) is irreducible and $V=\operatorname{span}\Delta$ then $Z_W(V)=\{cI : c\neq 0\}$

Pf Consider $A\in Z_W(V)$.

A, σ_α commute $\Rightarrow$ $A\sigma_\alpha\alpha=\sigma_\alpha A\alpha$

$-A\alpha=\sigma_\alpha A\alpha$

$A\alpha=c_\alpha\alpha$

$\Rightarrow A\sigma_\alpha\beta=\sigma_\alpha A\beta$

$A(\beta-\langle\beta,\alpha^\vee\rangle\alpha)=A\beta-\langle A\beta,\alpha^\vee\rangle\alpha$

$\langle\beta,\alpha^\vee\rangle c_\alpha\alpha=\langle c_\beta\beta,\alpha^\vee\rangle\alpha$

$c_\alpha=c_\beta$ or $\langle\alpha,\beta\rangle=0$

For $\alpha,\beta\in\Delta$, $\underset{c_\alpha=c_\beta}{\circ\!-\!\circ}$ for all edges of the Coxeter diagram. Since the dgm is connected, all c_α are equal.

$\to A\alpha=c\alpha$ for all $\alpha\in\Delta$

$\to Av=cv$ for all $v\in V$

$\to A=cI$.

Theorem W finite $\Leftrightarrow$ $\langle , \rangle$ pos. def.

Pf. $\Leftarrow$: done

$\Rightarrow$: Enough to do for W irred.

By Lemma 4, $Z_W(V) = \{cI : c \neq 0\}$

So then $\rho : W \to GL(V)$, the geom. repn., is irreducible.

Now, $\langle , \rangle$ is W-invariant ($\langle u,v\rangle = \langle w\cdot u, w \cdot v\rangle$)

$\langle , \rangle$ is non-degenerate

(V is W-invariant so $V^\perp$ is W-invariant so $V^\perp = V$ or 0)

So $\langle , \rangle$ is the unique W-invariant non-degenerate form on V. But the pos def form of Lemma 1 is also it $\to$ $\langle , \rangle$ is pos def ∎

Next: Classify the finite Coxeter gps by classifying the pos def $\langle , \rangle$

Recall: Γ Coxeter graph on $\{1, \dots, n\}$ $\to$ $A_\Gamma = \left[-\cos\frac{\pi}{m_{ij}}\right]_{1 \le i,j \le n}$

$\langle x, y \rangle = x^T A y$

$\langle x, x \rangle = x^T A x = \sum_{i,j=1}^{n} a_{ij} x_i x_j$

If $\langle , \rangle$ pos def, call A pos. def.

Lemma A symmetric matrix A is pos. def

$\Updownarrow$

its eigenvalues are >0.

Pf $\Downarrow$ Let λ be eigenvalue, v be eigenvector

$\to v^T A v = v^T \lambda v = \lambda \underbrace{(v^T v)}_{>0}$ $\Rightarrow \lambda > 0$

$\Uparrow$ A symmetric $\to$ diagonalizable, orthonormal basis of eigenvectors; $\lambda_i \in \mathbb{R}$

$A v_i = \lambda_i v_i$

"Spectral Theorem" (easy induction)

Then $v^T A v = (\sum c_i v_i)^T (\sum c_i \lambda_i v_i)$

$= \sum_{i=1}^{n} \lambda_i c_i^2 > 0.$ ∎

Prop. (Sylvester) A symmetric matrix is pos def

$\Updownarrow$

its principal minors are >0

Ppal minors: