Theorem. W is finite $\Updownarrow$ $\langle , \rangle$ is positive definite

$\Uparrow$: (Assume some topology for a bit.)

Note $W \subset GL(V)$
$\downarrow$
$n \times n$ invertible matrices $\subset \mathbb{R}^{n^2}$

Claim. W is discrete.
(No $w \in W$ such that every nbhd of W in $\mathbb{R}^{n^2}$ contains ∞-ly many points of W)

Pf. Remember W acts on Tits cone
(one chamber per $w \in W$)

Take $x \in D$
Let $U = \{A \in GL(V) \mid Ax \in wD\}$
The only point of W here is w. ■

Now if $\langle , \rangle$ is pos. def then it is Euclidean so W acts by real reflections,
so $W \subset O(V)$ = orthogonal gp.
$\downarrow$
$n \times n$ matrices with $A^{-1} = A^T$

Now $O(V) \subset \mathbb{R}^{n^2}$ is:

- closed because $O(V)$ is cut out by algebraic equations
- bounded because orthogonal matrices have orthonormal columns, so they have length 1.

So $O(V)$ is compact.

If $W \subset O(V)$ was infinite, it would have an accumulation point ■

For the other direction we need a bit of representation theory.

Def A <u>representation</u> of a group G is a homomorphism $\rho: G \to GL(V)$ for some vector space V.

(A way of rep'ing G as a group of invertible linear transfs (or invertible matrices once we choose a basis for V).)

Ex: $\rho: S_3 \to GL(\mathbb{C}^3)$ $\quad \rho(231) = \boxed{\begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}}$, etc

Given G finite $\rho: G \to GL(U)$

Lemma 1. There is a pos. def. bilinear form on V which is invariant under the action of G. $(\langle u,v\rangle = \langle g\cdot u, g\cdot v\rangle)$

Pf Take any pos. def. $(,)$ and "average" it over G:

$$\langle u,v\rangle = \sum_{g\in G} (g\cdot u, g\cdot v) \qquad \blacksquare$$

Def $G \to GL(V)$ is irreducible if V has no proper subspace $U \subset V$ invariant under G. $(GU \subseteq U)$

Def $\rho_i: G \to GL(U_i) \Rightarrow \rho_1 \oplus \rho_2 : G \to GL(U_1 \oplus U_2)$

Lemma 2. If G is finite then every rep. of G is a $\oplus$ of irreducibles

Pf Sup $G \to GL(V)$ is not irred - say $U \subset V$ is G-invariant

Claim: $U^\perp = \{v \in V \mid \langle u,v\rangle = 0 \text{ for all } u \in U\}$ is also G-invariant

Pf. $v \in U^\perp \to \langle gv, u\rangle = 0$ for $u \in U^\perp$

$$\langle v, \underbrace{g^{-1}u}_{\text{in } U}\rangle = 0 \quad \checkmark \qquad \blacksquare$$

So $\rho = \rho|_U \oplus \rho|_{U^\perp}$ and then induct $\blacksquare$

Coords: Can choose a basis so $V = V_1 \oplus \cdots \oplus V_k$ and

$$\rho(g) = \begin{array}{c} \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} \\ \begin{bmatrix} \rho_1(g) & & 0 \\ & \rho_2(g) & \\ 0 & & \rho_3(g) \end{bmatrix} \end{array}$$

Def For $\rho: G \to GL(U)$ let the centralizer of G be $Z_G(U) = \{A \in GL(U) : \forall g \in G \; Ag = gA\}$

Note. $\begin{bmatrix} aI_1 & & 0 \\ & bI_2 & \\ 0 & & cI_3 \end{bmatrix}$ is in $Z_G(U)$.

So if $Z_G(U) = \{cI : c \neq 0\}$ then $\rho: G \to GL(U)$ is irreducible $\longrightarrow$

Back to example:

In example, the line $L = \{(x,x,x)\}$ is invariant
$L^\perp = \{(x,y,z) \mid x+y+z = 0\}$ also

In basis $\underbrace{(1,1,1)}_{L}, \underbrace{(-1,1,0), (-1,0,1)}_{L^\perp}$, all $\rho = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\rho_2} \end{bmatrix}$

and $\rho = \rho_1 \oplus \rho_2$.