Our next goal:

CLASSIFYING FINITE COXETER GROUPS

(W,S) Coxeter gp

$S = \{s_1, \dots, s_n\}$

$m_{ij} \to$ Coxeter matrix $\longrightarrow$
- Basis of V: $\Delta = \{\alpha_1, \dots, \alpha_n\}$
- Bilinear form $\langle \alpha_i, \alpha_j \rangle = -\cos\left(\frac{\pi}{m_{ij}}\right)$

Review of bilinear forms

Representing matrices: Let $e_1, \dots, e_n$ be a basis.

Suppose that $u = \sum_i u_i e_i$, $v = \sum_j v_j e_j$, then

$$\langle u,v \rangle = \sum_{i,j} u_i v_j \langle e_i, e_j \rangle$$

$$= [u_1 \cdots u_n] \boxed{\left[\langle e_i, e_j \rangle\right]_{1\le i,j\le n}} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = U^T E V$$

"representing matrix"

So in coordinates, $\langle u,v \rangle = U^T E V$

If $f_1, \dots, f_n$ is a different basis, say

$$\begin{bmatrix} | & & | \\ f_1 & \cdots & f_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{bmatrix} \boxed{M}$$

then $u = \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix} U = \begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix} M^{-1} U$

So u has coord. $M^{-1}U$ wrt $\{f_i\}$
v $M^{-1}V$

Also check:

$$F = \left[\langle f_i, f_j \rangle\right]_{i,j} = M^T \left[\langle e_i, e_j \rangle\right]_{i,j} M = M^T E M$$

So now

$$\langle u,v \rangle = (M^{-1}U)^T (M^T E M)(M^{-1}V) = U^T E V \checkmark$$

new representing matrix.

If $\langle \, , \rangle$ is symmetric then E is symmetric, so it can be diagonalized:

$$M^{-1} E M = D \quad \text{for } M \text{ orthogonal } (M^{-1} = M^T)$$

diagonal

A bilinear form $\langle \, , \rangle$ is positive definite if $\langle v,v \rangle > 0$ for all $v \neq 0$.

Claim A symmetric $\langle \, , \rangle$ is positive definite
$\Updownarrow$
It is the Euclidean inner product wrt some basis.

$\Uparrow$ Clear $\Downarrow$: diagonalize it $\to$ get $D = \begin{bmatrix} >0 & & 0 \\ & >0 & \\ 0 & & >0 \end{bmatrix}$
so this corresponds to an orthogonal basis. $\boxtimes$

Theorem. W is finite

$\Updownarrow$

$\langle , \rangle$ is positive definite

$\Uparrow$: (Assume some topology for a bit.)

Note $W \subset GL(V)$
$\downarrow$
$n \times n$ invertible matrices $\subset \mathbb{R}^{n^2}$

Claim. W is discrete.
(No $w \in W$ such that every nbhd of W in $\mathbb{R}^{n^2}$ contains ∞-ly many points of W)

Pf. Remember W acts on Tits cone (one chamber per $w \in W$)

Take $x \in D$
Let $U = \{A \in GL(V) \mid Ax \in wD\}$
The only point of W here is w. ∎

Now if $\langle , \rangle$ is pos. def then it is Euclidean so W acts by real reflections, so $W \subset O(V)$ = orthogonal gp.
$\downarrow$
$n \times n$ matrices with $A^{-1} = A^T$

Now $O(V) \subset \mathbb{R}^{n^2}$ is:

- closed because $O(V)$ is cut out by algebraic equations
- bounded because orthogonal matrices have orthonormal columns, so they have length 1.

So $O(V)$ is compact.

If $W \subset O(V)$ was infinite, it would have an accumulation point ∎

For the other direction we need a bit of representation theory.

Def A *representation* of a group G is a homomorphism $\rho: G \to GL(V)$ for some vector space V.

(A way of rep'ing G as a group of invertible linear transfs (or invertible matrices once we choose a basis for V).)