Necessity follows from $\frac{a_{i_1 i_2}}{a_{i_2 i_1}} = \frac{\|\alpha_{i_1}\|}{\|\alpha_{i_2}\|}$

Sufficiency sketch: Dynkin diag → Coxeter diag → Cox gp → → root system → renormalize using to make crystall. ☒

Theorem A Coxeter system (W,S) is the reflection gp of a crystallographic root system

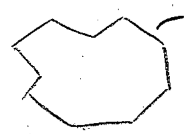
$\Updownarrow$

each $m_{ij} \in \{2,3,4,6,\infty\}$, and in each cycle of the Coxeter graph with no ∞ there is an even # of 4s and an even # of 6s.

Necessity:

a_{ij} a_{ji}	→	m_{ij}	$\|\alpha_i\|/\|\alpha_j\|$
o o o o		• •	
1 —— 1		——	1
1 ⇐ 2		$\overset{4}{——}$	$\sqrt{2}$
1 ⇚ 3		$\overset{6}{——}$	$\sqrt{3}$
a —— b $ab>4$		$\overset{\infty}{——}$	$\sqrt{a}/\sqrt{b}$

In an ∞-free cycle,



$a_{ij}/a_{ji} = 1, \sqrt{2}$, or $\sqrt{3}$

$\sqrt{2}^a \sqrt{3}^b = 1$

$\Rightarrow a,b$ even ✓

Sufficiency; given a graph, one needs to orient it so $\left\{\begin{array}{l} \#(\leftarrow) = \#(\rightarrow) \\ \#(\Leftarrow) = \#(\Rightarrow) \end{array}\right\}$ in each ∞-free cycle.

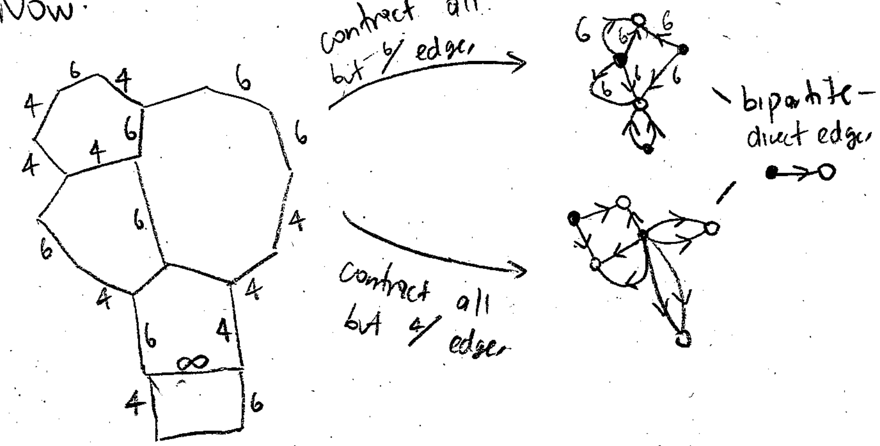
Def A graph is *bipartite* if one can split $V = V_1 \sqcup V_2$ so all edges go from V_1 to V_2

$\boxed{\text{Prop } G \text{ is bipartite} \Leftrightarrow \text{every cycle is even}}$

Pf. $\Rightarrow$: cycles go $V_1 - V_2 - V_1 - V_2 - \dots - V_1$.

$\Leftarrow$: Start at $v \in G$. Walk around the graph coloring the vertices alternatingly black and white.
If I repeat vertices, there are $2k$ edges in between so they get the same color.
Repeat until G is colored. ☒

Now:



Give = and ≡ edges these orientations
$\Rightarrow$ prod $=1$ along ∞-free cycles.
If I have an ∞ edge, choose $\sqrt{p}/\sqrt{q}$ freely for prod $=1$.
(If it is on two cycles, they will give the same $\sqrt{p}/\sqrt{q}$.) ☒