Prop Let (Φ, Δ) be crystallographic, and (W,S) be the Cox. sys. with Coxeter matrix m. Then

$$m_{ij} \in \{2, 3, 4, 6, \infty\}$$
$$\updownarrow \ \updownarrow \ \updownarrow \ \updownarrow \ \updownarrow$$
$$a_{ij}a_{ji} \in \{0, 1, 2, 3, \geq 4\}$$

Pf Consider $s_i \neq s_j$.

If they commute, then $\langle \alpha_i, \alpha_j \rangle = 0$ $\quad m_{ij} = 2$

Sup $m_{ij} \geq 3$. Let $\langle \alpha_i, \alpha_j^\vee \rangle = a$,
$\langle \alpha_i^\vee, \alpha_j \rangle = b$.

On $\mathrm{span}(\alpha_i, \alpha_j)$,

$$s_i = \begin{bmatrix} -1 & -b \\ 0 & 1 \end{bmatrix} \quad s_j = \begin{bmatrix} 1 & 0 \\ -a & -1 \end{bmatrix}$$

so

$$s_i s_j = \begin{bmatrix} ab-1 & b \\ -a & -1 \end{bmatrix} \qquad \begin{matrix} \det = 1 \\ \mathrm{tr} = ab-2 \end{matrix}$$

Note $\dim \alpha_i^\perp, \alpha_j^\perp \geq n-1$

$\rightarrow \dim \alpha_i^\perp \cap \alpha_j^\perp \geq n-2$ and $s_i s_j$ fixes it

$\rightarrow$ At least $n-2$ eigenvectors with $\lambda = 1$

$\rightarrow$ Other two are $\lambda, 1/\lambda$ of

$$\begin{bmatrix} ab-1 & b \\ -a & -1 \end{bmatrix}$$

- If $m_{ij} < \infty$:

Then $\lambda^{m_{ij}} = 1 \rightarrow \lambda, \frac{1}{\lambda} = e^{\frac{\pm 2k\pi i}{m_{ij}}}$ $\qquad 0 \leq k < m_{ij}$

If they are 1, then

- if diagonalizable, then $s_i s_j = e \rightarrow m_{ij} = 1$
- if not, then

$$s_i s_j v_1 = v_1$$
$$s_i s_j v_2 = v_2 + k v_1 \qquad k \neq 0$$
$$(s_i s_j)^{m_{ij}} v_2 = v_2 + k m_{ij} v_1 \neq v_2$$

So they are not 1

$$\rightarrow \mathrm{tr}\begin{bmatrix} ab-1 & b \\ -a & -1 \end{bmatrix} = \underbrace{ab-2}_{\mathbb{Z}} = 2\cos\left(\frac{2k\pi}{m_{ij}}\right)$$

$\downarrow$

$\cancel{-2}, -1, 0, 1, \cancel{2}$ $\quad (\lambda \neq 1)$

$(ab > 0)$

$ab = 1, 2, 3$ $\qquad \cos\left(\frac{2k\pi}{m_{ij}}\right) = -\frac{1}{2}, 0, \frac{1}{2}$

$= \cos\left(\frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}\right)$

$\downarrow$

$m_{ij} = 3, 4, 6$

- If $m_{ij} = \infty$, sup $ab \leq 3$

On $\mathrm{span}(\alpha_i, \alpha_j)$ $\begin{bmatrix} ab-1 & b \\ -a & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}$

$\qquad \lambda^2 = \lambda + 1 \qquad \lambda^2 + 1 \qquad \lambda^2 + \lambda + 1$

order: $\quad 3 \qquad 4 \qquad 6$

On $\alpha_i^\perp \cap \alpha_j^\perp$ it is the identity $\longrightarrow |s_i s_j| = 3, 4$ or 6 $\Rightarrow\Leftarrow$ $\blacksquare$