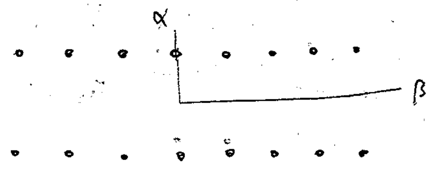
$V=\mathbb{R}^2=\operatorname{span}\{\alpha,\beta\}$
$\langle\alpha,\alpha\rangle=1 \quad \langle\beta,\beta\rangle=0$
$\langle\alpha,\beta\rangle=0$



$$\Phi=\{\pm\alpha+n\beta \mid n\in\mathbb{Z}\}$$
$$\Delta=\{\alpha,\beta-\alpha\}$$

(R1) $\langle\pm\alpha+n\beta,\pm\alpha+n\beta\rangle=1$

(R4) $\sigma_\alpha(\alpha+n\beta)=\alpha+n\beta-2\dfrac{\langle\alpha+n\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha=-\alpha+n\beta$

$\sigma_{\beta-\alpha}(\alpha+n\beta)=\alpha+n\beta-2\dfrac{\langle\alpha+n\beta,\beta-\alpha\rangle}{\langle\beta-\alpha,\beta-\alpha\rangle}(\beta-\alpha)$

$=\alpha+n\beta+2(\beta-\alpha)=-\alpha+(n+2)\beta$

Check: $\langle\sigma_\alpha,\sigma_{\beta-\alpha}\rangle=I_2(\infty)$

$V=\mathbb{R}^n$
$\langle,\rangle=$ Euclidean

$(0\,0\cdots1\cdots-1\cdots0)$

$$\Phi=\{e_i-e_j \mid 1\le i\ne j\le n\}$$
$$\Delta=\{e_{i+1}-e_i \mid 1\le i\le n-1\}$$

(R4): $\sigma_{e_{i+1}-e_i}(v)=(v_1,\dots,v_{i+1},v_i,\dots,v_n)$
still one 1, one -1

(R5): $\underset{i>j}{e_i-e_j}=(e_i-e_{i-1})+\dots+(e_{j+1}-e_j)$

Note: $\langle\sigma_{e_{i+1}-e_i}\rangle=S_n$

$V=\mathbb{R}^n$
$\langle,\rangle=$ Euclidean

goal: signed permutations

need $e_{i+1}-e_i$ → swap consec. cards
need e_1 → turn over top card
↳ $\sigma_1(e_2-e_1)=e_1+e_2$ → need e_i+e_j
$\sigma_{e_2-e_1}(e_1)=e_2$ → need e_i

$$\Phi=\{\pm e_i\pm e_j,\ \pm e_i\}$$
$1\le i,j\le n \quad 1\le i\le n$

$$\Delta=\{e_1,e_2-e_1,e_3-e_2,\dots,e_n-e_{n-1}\}$$

Check: $\langle\sigma_\alpha:\alpha\in\Delta\rangle=S_n^B=:B_n$

Some simple lemmas:

Prop. Take $\varphi\in GL(V)$ with $\langle\varphi v,\varphi v\rangle=\langle v,v\rangle$
Take $\alpha,\beta\in V$ with $\langle\alpha,\alpha\rangle,\langle\beta,\beta\rangle\ne0$.
- $\varphi\sigma_\alpha\varphi^{-1}=\sigma_{\varphi\alpha}$
- $\sigma_\alpha=\sigma_\beta \iff \alpha=c\beta$ for $c\ne0$
- $\sigma_\alpha,\sigma_\beta$ commute $\iff \langle\alpha,\beta\rangle=0$ or $\sigma_\alpha=\sigma_\beta$.

- $\varphi\sigma_\alpha\varphi^{-1}(v)=\varphi\left(\varphi^{-1}(v)-\dfrac{2\langle\varphi^{-1}v,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha\right)$

$=v-2\dfrac{\langle v,\varphi\alpha\rangle}{\langle\varphi\alpha,\varphi\alpha\rangle}\varphi\alpha=\sigma_{\varphi\alpha}(v)$

- $\Rightarrow$: $\sigma_\alpha=\sigma_\beta \Rightarrow \dfrac{\langle\alpha,v\rangle}{\langle\alpha,\alpha\rangle}\alpha=\dfrac{\langle\beta,v\rangle}{\langle\beta,\beta\rangle}\beta\ \forall v \Rightarrow \alpha=c\beta$

- $\sigma_\alpha \sigma_\beta v = v - 2\frac{\langle v,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha - 2\frac{\langle v,\beta\rangle}{\langle\beta,\beta\rangle}\beta + 4\frac{\langle v,\beta\rangle\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle}\alpha$

$\sigma_\alpha\sigma_\beta v = \sigma_\beta\sigma_\alpha v$ all v $\rightarrow$ $\langle\alpha,\beta\rangle = 0$

or $\alpha = c\beta$ ☒

<u>Prop</u>
- If $\beta \in \Phi$ then $\langle\beta,\beta\rangle > 0$, $\sigma_\beta \in W$, $-\beta \in \Phi$
- For $\langle\alpha,\beta\rangle \in \Delta$, $\langle\alpha,\beta\rangle \le 0$

<u>Pf</u>
- $\beta = w\alpha$ $\quad w \in W$ $\alpha \in \Delta$
  - ↳ $\langle\beta,\beta\rangle = \langle\alpha,\alpha\rangle > 0$
  - ↳ $\sigma_\beta = w\sigma_\alpha w^{-1} \in W$
  - ↳ $-\beta = \sigma_\beta\beta \in \Phi$
- $\sigma_\alpha\beta = \beta - 2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha$ should not have >0 and <0 coeffs. ☒

<u>Lemma</u> For $\alpha \in \Delta$, σ_α permutes $\Phi^+ - \{\alpha\}$.

<u>Pf</u>. For $\beta \in \Phi^+$, $\beta \neq \alpha$ $\quad \sigma_\alpha\beta = \beta - \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha$ has a pos. coeff. and α is not a multiple of β, so $\sigma_\alpha\beta \in \Phi^+$. ☒

Theorem

(Φ,Δ) root system $\Rightarrow$ (W,S) Coxeter group, $\quad S = \{\sigma_\alpha : \alpha \in \Delta\}$

<u>Pf</u> <u>Claim</u>. $w = \sigma_1 \cdots \sigma_n$ reduced, $\alpha \in \Delta$. TFAE:

1. $w\alpha < 0$
2. $\ell(w\sigma_\alpha) < \ell(w)$
3. $w = \sigma_1 \cdots \hat{\sigma}_i \cdots \sigma_n \sigma_\alpha$ some i.

By ② ⇒ ③ and Matsumoto, we get the thm.

Pf of Claim:

<u>① ⇒ ③</u>: $\alpha > 0, \ \ldots, \ \sigma_1 \cdots \sigma_n \alpha < 0$

So some $\sigma_i \cdots \sigma_n\alpha > 0$, $\sigma_{i+1} \cdots \sigma_n \alpha < 0$

By lemma, $\sigma_{i+1} \cdots \sigma_n \alpha = \alpha_i$

$\alpha = \sigma_n \cdots \sigma_{i+1}\alpha_i$

$\sigma_\alpha = \sigma_n \cdots \sigma_{i+1}\sigma_i\sigma_{i+1} \cdots \sigma_n$

$w\sigma_\alpha = \sigma_1 \cdots \hat{\sigma}_i \cdots \sigma_n$ ✓

<u>③ ⇒ ②</u>: $\ell(w\sigma_\alpha) \le n-1 < \ell(w)$.

<u>② ⇒ ①</u>: If $w\alpha > 0$, we'd have

$w\sigma_\alpha\alpha < 0 \overset{1\to 2}{\Longrightarrow} \ell(w\sigma_\alpha\sigma_\alpha) < \ell(w\sigma_\alpha)$ ☒