Root Systems

V - fin dim $\mathbb{R}$-vector space

$\langle\, , \rangle$ - symm. bilinear form

If $\alpha \in V$, $\langle \alpha, \alpha \rangle \neq 0$ $\rightarrow$ $\sigma_\alpha(v) = v - 2\dfrac{\langle v, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha$

"reflection"

Props: • $\sigma_\alpha^2 = 1$ (involution)

• $\langle \sigma_\alpha u, \sigma_\alpha v\rangle = \langle u, v\rangle$ (isometry)

Def A root system is a pair (Φ, Δ) where $\underset{\text{simple roots}}{\Delta} \subseteq \underset{\text{roots}}{\Phi} \subseteq V$ are such that:

(R1) $\langle \alpha, \alpha \rangle > 0$ for $\alpha \in \Delta$

(R2) Δ is lin. indep.

(R3) $\beta, c\beta \in \Phi \rightarrow c = \pm 1$

(R4) $\Phi = W\Delta$ where $W = \langle \sigma_\alpha : \alpha \in \Delta \rangle$

(R5) If $\beta \in \Phi$ then β is a non-neg or non-pos lin comb of Δ.

We know

(W,S) Coxeter gp $\rightarrow$ (Φ, Δ) root system

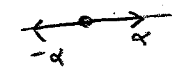
Now we will show

$(\underset{\langle \sigma_\alpha \rangle}{W}, \underset{\{\sigma_\alpha\}}{S})$ Coxeter gp $\leftarrow$ (Φ, Δ) root system

Some examples

$\boxed{V = \mathbb{R}^0}$ - empty root system

$\boxed{V = \mathbb{R}^1}$ $\Phi = \{\alpha, -\alpha\}$, $\Delta = \{\alpha\}$

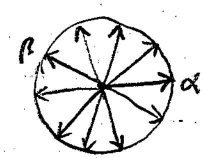


$\boxed{V = \mathbb{R}^2,\ \langle , \rangle \text{ Euclidean}}$

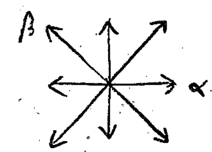
$\langle , \rangle$ = Euclidean

Φ = 2m unit vectors at equal angles

$\Delta = \{\alpha, \beta\}$



If m even, there are two orbits of roots, so can give them different lengths

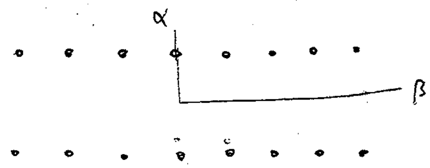


Note: $\langle \sigma_\alpha, \sigma_\beta \rangle = I_2(m)$ — dihedral gp of order 2m

$V = \mathbb{R}^2 = \mathrm{span}\{\alpha, \beta\}$

$\langle \alpha, \alpha \rangle = 1 \quad \langle \beta, \beta \rangle = 0$

$\langle \alpha, \beta \rangle = 0$



$$\Phi = \{\pm\alpha + n\beta \mid n \in \mathbb{Z}\}$$
$$\Delta = \{\alpha, \beta - \alpha\}$$

(R1) $\langle \pm\alpha + n\beta, \pm\alpha + n\beta \rangle = 1$

(R4) $\sigma_\alpha(\alpha + n\beta) = \alpha + n\beta - 2\frac{\langle \alpha + n\beta, \alpha\rangle}{\langle \alpha, \alpha \rangle}\alpha = -\alpha + n\beta$

$$\sigma_{\beta-\alpha}(\alpha + n\beta) = -\alpha + n\beta - 2\frac{\langle \alpha + n\beta, \beta - \alpha\rangle}{\langle \beta-\alpha, \beta-\alpha\rangle}(\beta - \alpha)$$
$$= \alpha + n\beta + 2(\beta - \alpha) = -\alpha + (n+2)\beta$$

Check: $\langle \sigma_\alpha, \sigma_{\beta-\alpha} \rangle = I_2(\infty)$

$V = \mathbb{R}^n$

$\langle , \rangle =$ Euclidean

$(0 0 \ldots 1 \ldots -1 \ldots 0)$

$$\Phi = \{e_i - e_j \mid 1 \le i \ne j \le n\}$$
$$\Delta = \{e_{i+1} - e_i \mid 1 \le i \le n-1\}$$

(R4): $\sigma_{e_{i+1}-e_i}(v) = (v_1, \ldots, v_{i+1}, v_i, \ldots, v_n)$

still one 1, one -1

(R5): $\underset{i>j}{e_i - e_j} = (e_i - e_{i-1}) + \ldots + (e_{j+1} - e_j)$

Note: $\langle \sigma_{e_{i+1}-e_i} \rangle = S_n$

$V = \mathbb{R}^n$

$\langle , \rangle =$ Euclidean

goal: signed permutations

need $e_{i+1} - e_i$ → swap consec. cards

need e_1 → turn over top card

↳ $\sigma_1(e_2 - e_1) = e_1 + e_2$ → need $e_i + e_j$

$\sigma_{e_2-e_1}(e_1) = e_2$ → need e_i

$$\Phi = \{\underset{1\le i,j \le n}{\pm e_i \pm e_j}, \underset{1 \le i \le n}{\pm e_i}\}$$

$$\Delta = \{e_1, e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1}\}$$

Check: $\langle \sigma_\alpha : \alpha \in \Delta \rangle = S_n^B =: B_n$