Difficulty: It is very hard to obtain info about a group from a presentation in terms of generators + relations.

"Word problem" (to tell whether a word is the identity) is undecidable - no algorithm can tell in general!

(Decidable for (W,S), but that takes work!)

We want a more concrete way of realizing the group.

Recall:

Cayley: Every G lies inside a symmetric group.

- $G \cong G'$ for G' a subgroup of S_G
- G "is" a group of permuts.

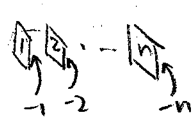
We want W as a group of signed permuts.

<u>Hyperoctahedral group S_n^B</u>

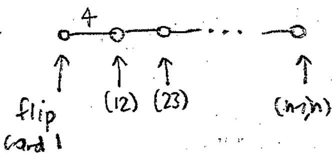
Signed permutations:

→ permuts of $\{1, 2, .., n, -1, -2, .., -n\}$ such that $\pi(-i) = -\pi(i)$

→ shuffles of



S_n^B = Coxeter group for



The "signed permutation rep'n" of (W,S)

Given: A Coxeter system (W,S)

Goal: Find a copy of it inside some S_T^B.

① Which T?

$$T = \{wsw^{-1} \mid w \in W, s \in S\} \quad \text{"reflections"}$$

② Where is (W,S) inside S_T^B?

Given $s \in S$, need $\pi_s \in S_T^B$.

" $w \in W$ " π_w "

- $\pi_s(t) = \begin{cases} sts & \text{if } s \neq t \\ -s & \text{if } s = t \end{cases}$

(Why a permutation?)

- $\pi_w(t) = \pi_{s_1} \pi_{s_2} \cdots \pi_{s_k}(t)$ for $w = s_1 \cdots s_k$

(Why doesn't it depend on $s_1, .., s_k$?)

③ Is this really a copy of (W,S)?

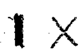
$\pi: W \longrightarrow S_T^B$ is an injective group homom.

So $W \cong \pi(W)$ ($\pi(W)$ subgroup of S_T^B)

Ex $(S_3, \{a,b\})$ $a=(12)$, $b=(23)$

$$T = \{\overset{a}{(12)}, \overset{b}{(23)}, \overset{c}{(13)}\} \qquad (c = aba = bab)$$

$\pi_a \to \pi_a(a) = -a \quad \pi_a(b) = aba = c \quad \pi_a(c) = aca = b$

gens
- $\pi_a = [-a, c, b] =$ 
- $\pi_b = [c, -b, a] =$ 

others $\pi_c = ?$ $\pi_a\pi_b\pi_a =$  $= (-b, -a, -c)$ $\quad \pi_b\pi_a\pi_b =$  $= (-b, -a, -c)$

They agree! Not obvious!

$$\left.\begin{array}{l} \pi_e = [a,b,c] \\ \left.\begin{array}{l}\pi_a = [-a,c,b] \\ \pi_b = [c,-b,a]\end{array}\right\} S' \\ \pi_{ab} = [b,-a,-c] \\ \pi_{ba} = [-c,a,-b] \\ \pi_c = [-b,-a,-c] \end{array}\right\} W'$$

$(W,S) \cong (W', S')$

Prop (W,S) = Coxeter system
T = reflections in W
The map $s \mapsto \pi_s$ extends uniquely to
$$\pi: W \to S_T^B$$
which is an injective group homom.

Pf By HW1, it suffices to show
(a) $\pi_s \in S_T^B$
(b) $(\pi_s \pi_{s'})^{m(s,s')} = e$
(c) π injective

Recall:
$\pi_s(t) = \pm sts$
($-$ iff $t = s$)

(a) $\pm sts = \pm sus \Rightarrow t = u$ ∎

(b) Note

$$\pi_{s_1}\pi_{s_2}\cdots\pi_{s_k} t = \pm s_1 s_2 \cdots s_k t s_k \cdots s_1$$

$\pm = (-1)^{n(s_1 s_2 \cdots s_k, t)}$, $n(s_1 s_2 \cdots s_k, t)$ = # of times that $t = s_1 s_2 \cdots s_i \cdots s_2 s_1$ $(1 \le i \le k)$

So
$$(\pi_s \pi_{s'})^m t = \pm (ss')^m t (ss')^m = \pm t$$

Check: $+t$ ✓ ∎

(c) Sup $\pi_u = \pi_v \Rightarrow \pi_w = e$ for $w = uv^{-1}$.
But let $w = s_1 \cdots s_k$ with k min.
Then $n(s_1 \cdots s_k, s_1) = 1$ (Or $s_1 \cdots s_i = s_2 \cdots s_{i-1}$)
So $\pi_w(s_1) = -ws_1w^{-1} \Rightarrow \pi_w \ne e$ ∎