<u>Counting walks in graphs</u> (to count reduced words)

G - digraph

$a_{uv} = \#$ of edges $u \to v$

↑

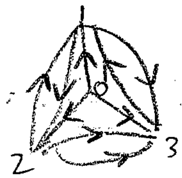
adjacency matrix

$P_{uv}(n) = \#$ of paths $\underbrace{u \to \cdots \to v}_{n \text{ edges}}$

Goal: Find $P_{uv}(n)$.

Ex

automaton for $^{\infty}\Delta^{\infty}_{\infty}$



$$A = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 & 0 \end{array}$$

Transfer-Matrix Method (statistical mechanics / Markov chains in prob.)

Note:

$$P_{uv}(n) = \sum_{w} a_{uw} P_{wv}(n-1)$$

$$\boxed{P(n)} = \boxed{A}\,\boxed{P(n-1)} = \boxed{A}^{n}$$

$(P(0) = I)$

> **Theorem** Fix u, v.
>
> $$\sum_{n \geq 0} P_{uv}(n) x^n = (-1)^{u+v} \frac{\det(I - xA; j, i)}{\det(I - xA)}$$
>
> del. row j col i

<u>Pf.</u> $\displaystyle\sum_{n\geq 0} P_{uv}(n) x^n = \sum_{n \geq 0} (A^n)_{uv} x^n$

$$= \Big(\sum_{n\geq 0} A^n x^n\Big)_{uv} = \big((I - Ax)^{-1}\big)_{uv} \quad \text{as claimed.} \ \blacksquare$$

In Ex,
$$\sum_{n\geq 0} P_{02}(n) x^n = \frac{\det\begin{vmatrix} \cancel{} & -x & -x & -x \\ \cancel{0} & 1 & -x & -x \\ \cancel{0} & \cancel{-x} & \cancel{1} & \cancel{-x} \\ \cancel{0} & -x & -x & 1 \end{vmatrix}}{\det\begin{vmatrix} 1 & -x & -x & -x \\ 0 & 1 & -x & -x \\ 0 & -x & 1 & -x \\ 0 & -x & -x & 1 \end{vmatrix}} = \frac{x^2 - x^3 + x^2 + x^3 + x^3 + x}{1 + x^3 + x^3 - x^2 - x^2 - x^2} = \frac{x + 2x^2 + x^3}{1 - 3x^2 - 2x^3}$$

$$= \frac{x(1+x)^2}{(1+x)^2(1-2x)} = \frac{x}{1-2x}$$

So if $r_n = \#$ of reduced words of length n,

$$r_n = \begin{cases} 1 & n = 0 \\ 3P_{02}(n) & n \geq 1 \end{cases}$$

$$\sum_{n\geq 0} r_n x^n = 1 + \frac{3x}{1-2x} = 1 + \sum_{n\geq 1} 3 \cdot 2^{n-1} x^n$$

$r_n = 3 \cdot 2^{n-1} \quad n \geq 1$

Not too impressive, but this is $^{\infty}\Delta^{\infty}_{\infty}$!

> <u>Corollary</u> $R_{(W,S)}(x) = \sum_{n \geq 0} r_n x^n$ is a rational function for all (W,S)

Often this is better than a formula for r_n.

$$R_{\Delta}(x) = \frac{(1+2q)(1+q^2)}{(1-q)(1-2q^2)} \ ; \ r_n = \_(\sqrt{2})^n + \_(-\sqrt{2})^n + \_$$

$1, \sqrt{2}, -\sqrt{2}$