Coxeter groups are automatic

A - set

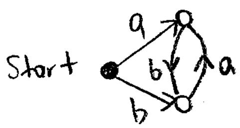
$\mathcal{L} \subseteq A^*$ - "formal language" (set of words on A)

An <u>automaton</u> is a finite digraph with a designated start vertex, and edges labelled by elts of A.

It <u>recognizes</u> the language

$\mathcal{L} = \{$ words obtained by reading the edge labels of a path starting at "start" $\}$

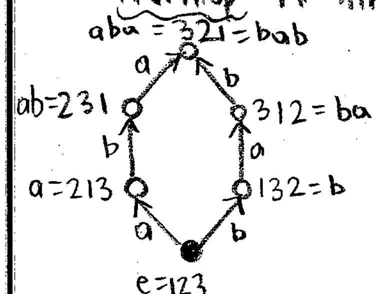
Such an $\mathcal{L}$ is <u>"regular"</u>



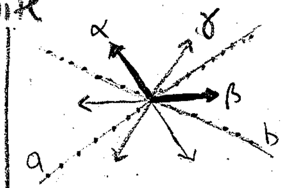
$\rightarrow \mathcal{L} = \{ \underbrace{aba\ldots}_{n}, \underbrace{bab\ldots}_{n} \mid n \geq 0 \}$

<u>Theorem</u> The language of reduced words in (W,S) is regular

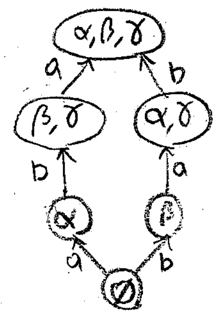
<u>Warmup</u>: W finite



right weak order



$\mathrm{Neg}(w) = \{ \gamma \in \Phi^+ \mid w\gamma \in \Phi^- \}$



$\mathrm{Neg}(w)$

<u>For $ws_i > w$</u>: If $\mathrm{Neg}(w) = \{\gamma_1, \ldots, \gamma_k\}$ $\quad k = \ell(w)$

then $\mathrm{Neg}(ws_i) = \{ s_i\gamma_1, \ldots, s_i\gamma_k, \alpha_i \}$ $\quad k+1 = \ell(ws_i)$

$= s_i \mathrm{Neg}(w) \cup \{\alpha_i\}$

We could do this for W infinite, but the graph would be infinite.

Instead, do this with:

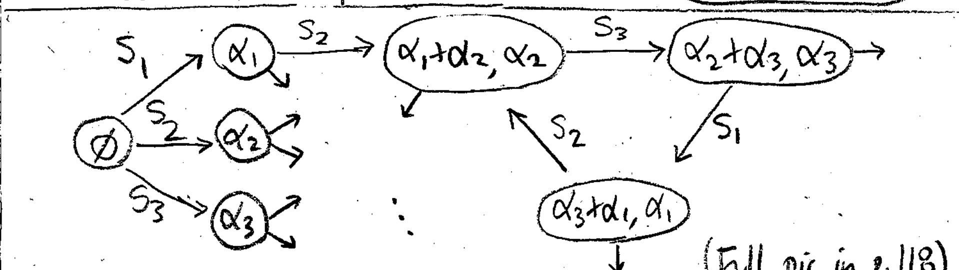
Let $\Sigma = \{\text{small roots}\} \subset \Phi^+$

Let $\mathrm{Neg}_\Sigma(w) = \{ \gamma \in \Sigma \mid w\gamma < 0 \}$

<u>Claim</u>. For $ws_i > w$,

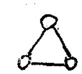
$$\mathrm{Neg}_\Sigma(ws_i) = \left(s_i \mathrm{Neg}_\Sigma(w) \cup \{\alpha_i\} \right) \cap \Sigma$$

Vertices: $\mathrm{Neg}_\Sigma(w)$ | Edges: $\alpha_i \notin S$: $S \xrightarrow{s_i} (s_i S \cup \{\alpha_i\}) \cap \Sigma$



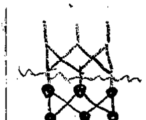
(Full pic in p.118)

$W = \tilde{A}_2$



$\Sigma = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_1+\alpha_3, \alpha_2+\alpha_3\}$

root poset:



$s_i\alpha_i = -\alpha_i$ $\qquad s_i(\alpha_i+\alpha_j) = \alpha_j$

$s_i\alpha_j = \alpha_i + \alpha_j$ $\qquad s_i(\alpha_j+\alpha_k) = 2\alpha_i + \alpha_j + \alpha_k$

Note: In example, $Neg_\Sigma(s_1s_2) = Neg_\Sigma(s_1s_2s_3s_1s_2)$

Question. What can we say about the map

$$w \mapsto Neg_\Sigma(w)$$

Image?

Fibers? (the sets $\{w\in W \mid Neg_\Sigma(w)=S\}$, $S\in\Sigma$)

There may be open questions.

∘ Claim → Theorem

Need: $s_1\cdots s_i$ reduced $\longleftrightarrow$ walk in graph

Induct on i, clear for $i=0$.

Sup true for i.

$s_1\cdots s_i s_{i+1}$ reduced $\longleftrightarrow$ $s_1\cdots s_i$ reduced
$s_1\cdots s_i\alpha_{i+1}>0$

$\longleftrightarrow$ walk $s_1,\dots,s_i$ and
$\alpha_{i+1}\notin$ last node

$\longleftrightarrow$ walk $s_1,\dots,s_i,s_{i+1}$ ✓ ■

∘ Proof of Claim

Lemma. Let $\alpha\in\Sigma$, $s_i\alpha\in\Phi^+\setminus\Sigma$.
Let $ws_i>w$. Then $ws_i\alpha>0$.

Pf

$s_i\alpha$
| long
α

We proved that in this situation $s_i\alpha$ dominates α_i.

Since $w\alpha_i>0 \to ws_i\alpha>0$. ■

Then the claim follows because

$$\begin{matrix}\alpha\in\Sigma\\ \alpha\in Neg(ws_i)\\ ws_i>w\end{matrix} \to s_i\alpha\notin\Phi^+\setminus\Sigma \to \begin{cases} s_i\alpha\in\Phi^- \to \alpha=\alpha_i \\ \text{or} \\ s_i\alpha\in\Sigma \to s_i\alpha\in Neg_\Sigma w\end{cases}$$

(the other inclusion is easy). ■

Note the subtlety:

all roots — Using $Neg(w)$ gave an infinite graph. — too much

$Neg(w) \updownarrow T_R(w)$

simple roots — Using $Neg_\Delta(w)$ would make it finite, but now $Neg_\Delta(ws_i)$ doesn't just depend on $Neg_\Delta(w)$, s_i — too little

$Neg_\Delta(w) \updownarrow D_R(w)$

small roots — Using $Neg_\Sigma(w)$ → gave a finite graph and

→ Can build $Neg_\Delta(ws_i)$ from $Neg_\Delta(w)$ and s_i only. — just right