Three nice properties:

① **Prop** The small roots are an order ideal in the root poset.

Pf. If β small and $\begin{matrix}\beta \\ | \\ \gamma\end{matrix}$ then this edge is short, and smallness propagates along short edges. $\blacksquare$

Lemma. If β dom γ then $dp(\beta) > dp(\gamma)$

Pf. Let $w\beta < 0$ with $\ell(w) = dp(\beta)$.

Let $w = s_a w'$ with $\ell(w) = \ell(w') + 1$

$w'\beta > 0$, $s_a w'\beta < 0 \longrightarrow w'\beta = \alpha_a$

Since β dom γ, $w\gamma < 0$.

If $w'\gamma > 0$ then $w'\gamma = \alpha_a$ so $\beta = \gamma$.

Thus $w'\gamma < 0 \longrightarrow dp(\gamma) \le \ell(w') = dp(\beta) - 1$. $\blacksquare$

Lemma If W is finite then $\beta \mapsto -w_0\beta$ is a depth-preserving, dom-reversing permut. of Φ^+.

Pf Clearly a permut.

dom: β dom $\gamma \Rightarrow$

$$\begin{aligned} w\beta < 0 &\longrightarrow w\gamma < 0 \\ w\beta > 0 &\longleftarrow w\gamma > 0 \\ ww_0\beta > 0 &\longleftarrow ww_0\gamma > 0 \end{aligned}$$

$-w_0\gamma$ dom $-w_0\beta$

depth Sup $w\beta < 0$ $\quad \ell(w) = dp\,\beta$

$ww_0(-w_0\beta) > 0$

$w_0 w w_0 (-w_0\beta) < 0$

Therefore $dp(-w_0\beta) \le \ell(w_0 w w_0)$
$= \ell(w) = dp(\beta)$

and this relation is "involutive" $\blacksquare$

② **Prop** If W is finite, every root is small.

Pf If β dom γ then $-w_0\gamma$ dom $-w_0\beta$

$$\begin{matrix} \downarrow & & \\ dp\,\beta > dp\,\gamma & \quad & dp(-w_0\gamma) > dp(-w_0\beta) \\ & & \parallel \qquad\qquad \parallel \\ & & dp\,\gamma \qquad\qquad dp\,\beta \end{matrix}$$ $\blacksquare$

Prop β dom $\gamma \iff dp(\beta) > dp(\gamma)$, $\langle \beta, \gamma \rangle \ge 1$

Key lemma:

- $-1 < \langle \alpha, \beta \rangle < 1 \Rightarrow$ subgp gen by t_α, t_β is D_k (finite)
- $\langle \alpha, \beta \rangle \le -1 \Rightarrow$ • subgp gen by t_α, t_β is D_∞.
  - all roots $(t_\alpha t_\beta)^n \alpha$ are distinct positive combins of α, β.

③ **Theorem** The number of small roots is finite.

Pf. See book.