Prop If $\gamma \geq \beta$ in the root poset,
$$\gamma = \sum c_i \alpha_i$$
$$\beta = \sum b_i \alpha_i$$
then each $c_i \geq b_i$

(Not converse: $101 \not< 211$ in $\Delta = \tilde{A}_2$)

Pf Enough for $\begin{tikzcd} \gamma \arrow[d, dash, "s_i"] \\ \beta \end{tikzcd}$ $\gamma = \beta - 2\langle \beta, \alpha_i \rangle \alpha_i$

Need $\langle \beta, \alpha_i \rangle < 0$

We have $\gamma = s_i \beta$ $\quad dp\,\gamma = dp\,\beta + 1$

$t_\gamma = s_i t_\beta s_i$ $\quad \ell(t_\gamma) = \ell(t_\beta) + 2$

So $\ell(s_i t_\beta s_i) > \ell(s_i t_\beta) \to (s_i t_\beta)\alpha_i > 0$

$\ell(t_\beta s_i) > \ell(t_\beta) \to t_\beta \alpha_i > 0$

Now
$$t_\beta \alpha_i = \alpha_i - 2\langle \beta, \alpha_i \rangle \beta > 0$$
This is positive. Now

- if $\langle \beta, \alpha_i \rangle = 0$ then $t_\beta \alpha_i = \alpha_i$, $s_i t_\beta \alpha_i = -\alpha_i < 0$
- if $\langle \beta, \alpha_i \rangle > 0$ then β must be a multiple of $\alpha_i \to \beta = \alpha_i \to t_\beta \alpha_i = s_i \alpha_i < 0$

$\square$

So coeffs. increase as we go up the poset

Small roots

Each edge is $\begin{tikzcd} \gamma \arrow[d, dash, "s_i"] \\ \beta \end{tikzcd}$ $\gamma = \beta + c\alpha_i$ $\quad c > 0$

Call the edge short if $c < 2$.

$(|\langle \gamma, \alpha_i \rangle| = |\langle \beta, \alpha_i \rangle| < 1)$

In S_n, all are short.

In $\tilde{A}_2$, have short and long.

Def A small root is one reachable from the bottom of the root poset by going up small edges.

Let $\beta, \gamma \in \Phi^+$

Say β dominates γ (β dom γ) if

$$w \cdot \beta < 0 \Rightarrow w \cdot \gamma < 0$$

Theorem A positive root is small if and only if it dominates no other root.

To prove this we need some lemmas.

Lemma. $\beta \in \Phi^+$ dominates $\alpha_i \iff \langle \beta, \alpha_i \rangle \geq 1$

Pf See book, lemma 4.7.4

<u>Lemma.</u> $\alpha \overset{\text{long}}{\text{—}} \beta \Rightarrow \alpha$ dominates someone

$\alpha \overset{\text{short}}{\text{—}} \beta \Rightarrow$ α dominates someone
$\Updownarrow$
β dominates someone

<u>Pf</u>

(a) $\alpha = s\beta \xrightarrow[\text{long}]{} \langle \alpha, \alpha_s \rangle \geq 1 \rightarrow \alpha$ dom α_s

(b) $\alpha = s\beta \xrightarrow[\text{short}]{} 0 > \langle \beta, \alpha_s \rangle > -1$

- If β dom γ then $\gamma \neq \alpha_s$

  So: $\underset{\alpha > 0}{\underset{\parallel}{s\beta}}$ dom $\underset{0}{\underset{\vee}{s\gamma}}$. (Return similarly) ☒

<u>Pf of Theorem</u>

- If α is short, take a path of short roots to α_i, which dominates noone ✓ $\quad \alpha \overset{\text{sh}}{\text{—}} \cdots \overset{\text{sh}}{\text{—}} \alpha_i$
- If α dominates noone, take a path to a simple root. Inductively, each edge must be short. ☒