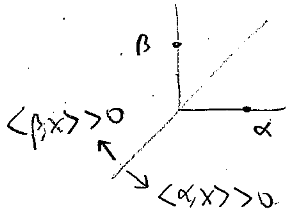
② $C = \{x \in V \mid \langle \alpha, x\rangle > 0, \langle \beta x\rangle > 0\} = \emptyset$

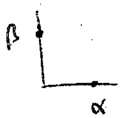


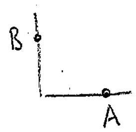
Instead of action of W on V, consider the "contragredient action" of W on V^*

V^* = vector space of linear fns on V.

$f \in V^* \mapsto wf$ characterized by

$wf(wv) = f(v)$ for all $v \in V$

Ex V: 

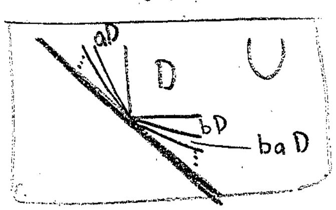
V^* 

$a\alpha = -\alpha$ $\quad b\alpha = \alpha + 2\beta$ $\quad \longrightarrow \quad aA = -A + 4B$ $\quad bA = A$

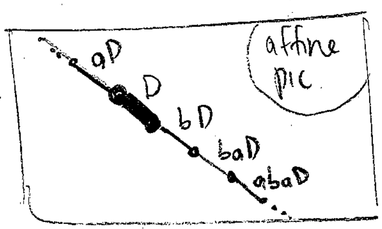
$a\beta = \beta + 2\alpha$ $\quad b\beta = -\beta$ $\quad\quad\quad aB = B$ $\quad bB = 4A - B$

$D = \{f \in V^* \mid f(\alpha) \geq 0, f(\beta) \geq 0\}$ fundam. domain

$U = \bigcup_{w \in W} wD$ $\quad \boxed{\text{Tits cone}}$



$\longrightarrow$

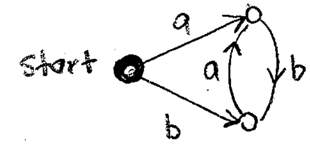


Our next goal: "Coxeter groups are automatic"

reduced words in W $\longleftrightarrow$ walks in a finite graph

Ex: 

reduced words: $\emptyset, a, b, ab, ba, aba, bab, abab, \ldots$



This will take some work:

root poset $\longrightarrow$ small roots $\longrightarrow$ automaticity

Root Poset

Recall $\quad W$ - Coxeter gp

$S = \{s_1, \ldots s_n\}$ gens

$\Delta = \{\alpha_1, \ldots, \alpha_n\}$ simple roots (basis for V)

$\langle \alpha_i, \alpha_j \rangle = -\cos(\pi/m_{ij})$

$\Phi = \{w \cdot \alpha_i \mid w \in W, 1 \leq i \leq n\}$ = roots

$= \Phi^+ \cup \Phi^-$

positive root $w\alpha_i$ $\updownarrow$ reflection ws_iw^{-1}

$w\alpha > 0$ $\Updownarrow$ $\ell(wt_\alpha) > \ell(w)$

w sends $\ell(w)$ roots from Φ^+ to Φ^-

Def The depth of a root $\beta>0$ is the min length of a w such that $w\beta<0$.

$$\boxed{\textbf{Prop}\quad dp(\beta)=\tfrac{1}{2}\left(\ell(t_\beta)+1\right)}$$

Pf Let $\beta = w\alpha_i$, $\ell(w)$ min

$t_\beta = w s_i w^{-1}$ $\qquad dp(\beta)=\ell(w)+1$?

Then

- $\underbrace{s_i w^{-1}}_{\ell(w)+1}(w\alpha_i)<0 \longrightarrow dp(\beta)\le \ell(w)+1 = \frac{1}{2}(\ell(t_\beta)+1)$
- Sup. $\underbrace{s_a s_b\cdots s_z}_{dp(\beta)}\beta<0$

  Then s_a sends $s_b\cdots s_z\beta<0$ to >0

  So $\alpha_a = s_b\cdots s_z\beta$

  $s_a = s_b\cdots s_z\, t\, s_z\cdots s_b$

  $t = s_z\cdots s_b s_a s_b\cdots s_z \longrightarrow \ell(t)\le 2dp(\beta)-1$ ▣

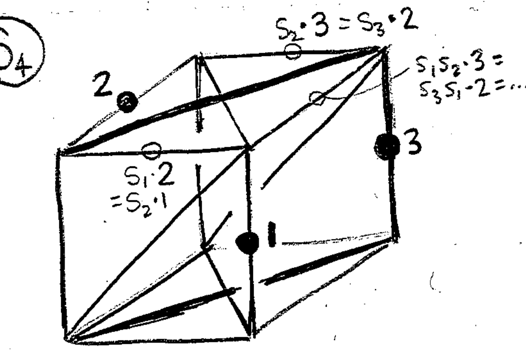
Def The root poset on Φ^+ is defined by

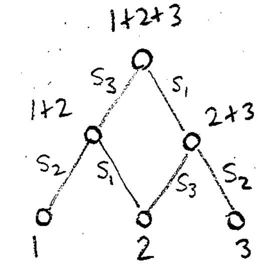
$$\beta \lhd \gamma \iff \gamma = s\beta,\ dp(\gamma)=dp(\beta)+1 \quad \text{for } s\in S$$

Note

- graded by depth
- edges labelled by S

S_4





The root poset for S_4.

The root poset of S_n is 

(Parenthesis:

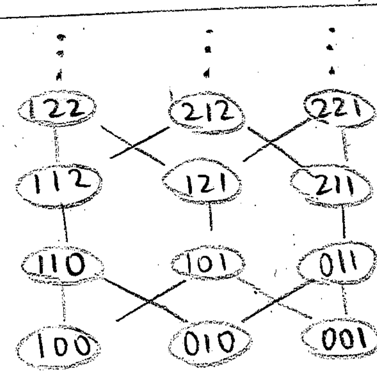
The number of antichains in the root poset of S_n $= C_{n-1} = \frac{1}{n}\binom{2(n-1)}{n-1}$

the Catalan number

(answer to ≥ 100 problems)

Can define similarly the "W-Catalan number."

100 problems? Not yet, but several already – this is an active area of research.)



Root poset for 

$(i\ j\ k) = i\alpha_1+j\alpha_2+k\alpha_3$

- $\langle\alpha_i,\alpha_j\rangle=-1/2 \quad \langle\alpha_i,\alpha_i\rangle=1$
- $s_1\alpha_1=-\alpha_1$

  $s_1\alpha_2=\alpha_1+\alpha_2$ $\quad$ etc.

  $s_1\alpha_3=\alpha_1+\alpha_3$