$\boxed{\text{Prop. For } I \subset S, \text{ the subgroup fixing } C_I \text{ is the parabolic subgroup } W_I}$

Pf.

○ W_I fixes C_I:

$s \in I, \lambda \in C_I$: $\alpha \in \Delta_I$

$s\lambda = \lambda - 2\langle \lambda, \alpha \rangle \alpha = \lambda$ ✓

○ Sup w fixes C_I.

Write $w = w^I w_I$. Need $w^I = e$.

$\ell(w^I s) > \ell(w^I)$ for $s \in I$

$w^I \cdot \alpha > 0$ for $\alpha \in \Delta_I$.

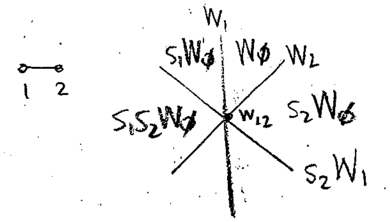
If $w^I \neq e$, find $\beta \in \Delta$ with

$w^I \beta < 0$ $\quad \beta > 0$ $\quad$ (so $\beta \notin \Delta_I$)

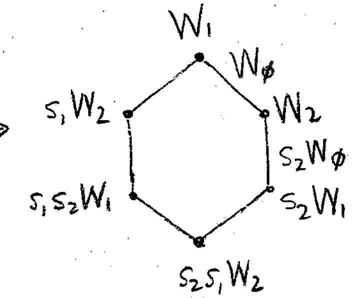
Take $\lambda \in C_I$, so $\left.\begin{matrix} w_I \lambda = \lambda \\ w\lambda = \lambda \end{matrix}\right\} w^I \lambda = \lambda$

$\beta \notin \Delta_I \rightarrow (\beta, \lambda) > 0 \rightarrow (\underbrace{w^I\beta}_{<0}, \underbrace{w^I\lambda}_{\lambda \in D}) > 0$ $\quad \Rightarrow\Leftarrow$ ∎

So I can think:



$\rightarrow$



(vertices = maxl proper cosets; faces : sets of vertices with non-empty $\cap$) $\leftarrow$ The Coxeter Complex

(a "simplicial complex")

What if W is infinite?

We can still get a Coxeter complex but a bit differently.

To see the problems:

$W = \overset{\infty}{\underset{a \quad b}{\circ\!-\!\!-\!\circ}}$ $\rightarrow \langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle = 1$

$\langle \alpha, \beta \rangle = -1$

① $\langle \alpha+\beta, \alpha \rangle = 1 - 1 = 0$

$\langle \alpha+\beta, \beta \rangle = -1 + 1 = 0$

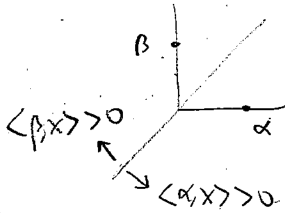
So $\langle \alpha+\beta, x \rangle = 0$ for all x but $\alpha+\beta \neq 0$!

$\langle \alpha+\beta, \alpha+\beta \rangle = 0$

Thus $\langle , \rangle$ is not positive definite

$\Rightarrow$ Not achievable in Euclidean space

② $C = \{x \in V \mid \langle \alpha, x\rangle > 0, \langle \beta x\rangle > 0\} = \emptyset$

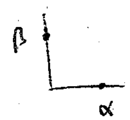


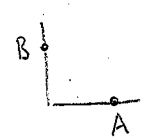
Instead of action of W on V, consider the "contragredient action" of W on V^*

V^* = vector space of linear fns on V.

$f \in V^* \mapsto wf$ characterized by

$wf(wv) = f(v)$ for all $v \in V$

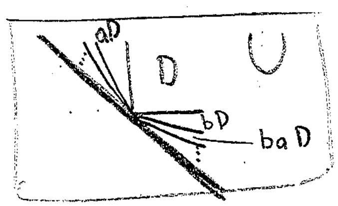
Ex V: 

V^* 

$a\alpha = -\alpha \quad b\alpha = \alpha + 2\beta$
$a\beta = \beta + 2\alpha \quad b\beta = -\beta$ $\longrightarrow$ $aA = -A + 4B \quad bA = A$
$aB = B \quad bB = 4A - B$

$D = \{f \in V^* \mid f(\alpha) \geq 0, f(\beta) \geq 0\}$ fundam. domain

$U = \bigcup_{w \in W} wD$ $\boxed{\text{Tits cone}}$



$\longrightarrow$