Now need $\ell(ws_i) > \ell(w) \Rightarrow w\alpha_i > 0$

Induct on $\ell(w)$. ($\ell(w)=0$ is obvious)

Take s with $ws < w$.

cant end in s, s_i — gen. by s, s_i

Let $J = \{s, s_i\}$ and use $W = W^J W_J$

$\Rightarrow w = w^J w_J$ • $\ell(w) = \ell(w^J) + \ell(w_J)$
• $w^J = $ min in coset wW_J

• $w\alpha_i = w^J \underbrace{w_J \alpha_i}$

→ Note $\ell(w_J s_i) > \ell(w_J)$ since

$\underbrace{ws_i}_{W} = \underbrace{w^J}_{W^J} \underbrace{w_J s_i}_{W_J}$ $\ell(ws_i) = \ell(w^J) + \ell(w_J s_i)$
$\vee$
$\ell(w) = \ell(w^J) + \ell(w_J)$

So by dihedral case

$w_J \alpha_i = a\alpha_i + b\alpha$ $a, b \geq 0$

→ So $w\alpha_i = w^J(a\alpha_i + b\alpha)$

Is $w^J\alpha_i > 0$? Is $w^J\alpha > 0$?

Note. $\ell(w) > \ell(w^J)$ since $w \notin W^J$ ($\ell(ws) < \ell(w)$)

— By induction, enough to show

$\ell(w^J s_i) > \ell(w^J)$ $\ell(w^J s) > \ell(w^J)$

But these follow since

$w^J s_i = w \underbrace{w_J^{-1} s_i} \in wW_J$

$w^J s \in wW_J$

∎

Corollary The geometric rep'n is faithful (injective)

Pf. $\sigma: W \to GL(V)$

Sup $\sigma(w) = \mathrm{id}$.

$\Rightarrow w\alpha_i = \alpha_i > 0$ for all α_i

$\Rightarrow \ell(ws_i) > \ell(w)$ for all s_i

$\Rightarrow w = e$. ∎

Prop.

(a) Let $s_i \in S$. Then s_i sends α_i to $-\alpha_i$ and permutes the other positive roots

(b) Let $w \in W$. Then

$\ell(w) = \#$ of pos. roots that w sends to neg. roots

Pf

(a) Clearly $s_i(\alpha_i) = -\alpha_i$. Now let $\alpha \in \Phi^+$, $\alpha \neq -\alpha_i$

Write

$\alpha = \sum_j c_j \alpha_j$ some $c_j > 0$ $j \neq i$

$s_i(\alpha) = \alpha - 2\langle \alpha, \alpha_i \rangle \alpha_i$

coeff of α_j is still $c_j > 0$

$\Rightarrow s_i(\alpha) > 0$.

So $s_i : \Phi^+ \setminus \{\alpha_i\} \to \Phi^+ \setminus \{\alpha_i\}$

Surj: $\alpha = s_i(s_i\alpha)$ Inj: $s_i\alpha = s_i\beta \Rightarrow s_is_i\alpha = s_is_i\beta$

(b) Induct on $\ell(w)$. $\ell(w)=1$ is ok by (a).

Sup true for length $< \ell(w)$.

Take s_i with $\ell(ws_i) < \ell(w)$

$(w\alpha_i < 0)$

$\Rightarrow \ell(ws_i)$ roots $\alpha > 0$ with $ws_i\alpha < 0$ → $\alpha \neq \alpha_i$: $ws_i\alpha_i = -w\alpha_i > 0$

→ $\ell(ws_i)$ roots $s_i\alpha > 0$ with $w(s_i\alpha) < 0$

↳ also not α_i

→ another one: $w\alpha_i < 0$

→ Clearly no others. $\Rightarrow \ell(ws_i)+1 = \ell(w)$ ∎

Roots, alg. reflections, geom. reflections

Consider how an alg. reflection $t = ws_iw^{-1}$ acts:

$$
\begin{aligned}
tv &= ws_iw^{-1}(v) \\
&= w[w^{-1}(v) - 2\langle w^{-1}(v), \alpha_i\rangle \alpha_i] \\
&= v - 2\langle w^{-1}(v), \alpha_i\rangle w\alpha_i \\
&= v - 2\langle v, w\alpha_i\rangle w\alpha_i
\end{aligned}
$$

$$\boxed{\text{action of alg refl: } \underset{\substack{\| \\ t \\ \text{reflection}}}{ws_iw^{-1}} = \text{geom refl. by root } \underset{\substack{\| \\ \alpha \\ \text{root}}}{w\alpha_i}}$$

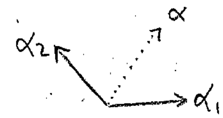
Lemma. $\langle wv_1, wv_2\rangle = \langle v_1, v_2\rangle$

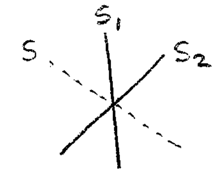
Pf. Compute, for each i, $\langle s_iv_1, s_iv_2\rangle = \langle v_1, v_2\rangle$.

So:

$$\boxed{\text{roots in } \Phi^+ \longleftrightarrow \text{alg. refls. in } W \xleftrightarrow{\text{HW4}} \text{geom refls. in geom. rep. of } W}$$

Ex: S_3





Prop. Let $w \in W$, $t \in T$ reflection, α = corr. root

$\ell(wt) > \ell(w) \Rightarrow w\alpha > 0$

$\ell(wt) < \ell(w) \Rightarrow w\alpha < 0$

Pf As in the case $t \in S$, enough to show one.

Sup $\ell(wt) < \ell(w)$

$w = s_1 s_2 \cdots s_k$

$t = \underbrace{s_k \cdots s_{i+1}}_{v} s_i \underbrace{s_{i+1} \cdots s_k}_{v^{-1}}$

↓

$\alpha = v\alpha_i$

$w\alpha = wv\alpha_i = \underbrace{s_1 s_2 \cdots s_i}\alpha_i < 0$

since s_i shortens $s_1 \cdots s_i$ ∎