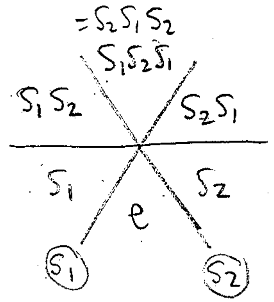
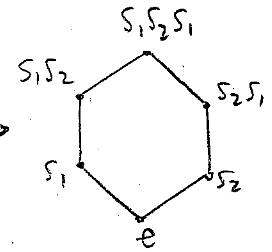
The weak order for S_3:

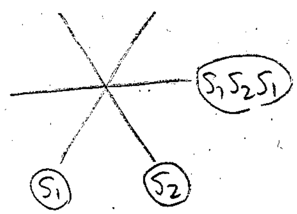
S_3 as group of reflections:

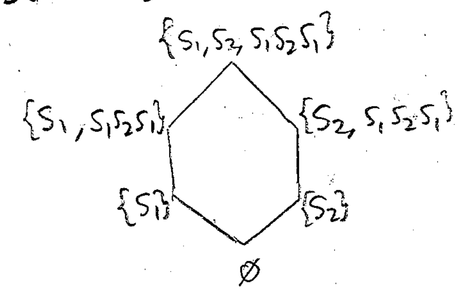


$\longrightarrow$



Recall $u < v \iff T_L(u) \subset T_L(v)$



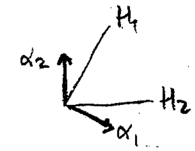


DICTIONARY

geometry	$\longleftrightarrow$	alg./comb.	
regions	$\longleftrightarrow$	group elements	W
walls of "home" region	$\longleftrightarrow$	generators	S
hyperplanes	$\longleftrightarrow$	reflections	T
hyps separating a region from "home"	$\longleftrightarrow$	reflections that shorten an element	$T_L(w)$
poset of regions by separation from "home"	$\longleftrightarrow$	weak order	

4. Geometric Representation of Coxeter Gps

Idea: I can represent S_3 as generated by two reflections at $60°$ angle.



s_1s_2 = rot. by $120°$
$(s_1s_2)^3 = e$

Can I do this in general for any W?

I can't always get the correct angle in Euclidean space, but I can change the way I measure angles!

> (W,S) Coxeter system, matrix m
> $S = \{s_1, \dots, s_n\}$
>
> - $V = \mathbb{R}$-vector space with basis $\alpha_1, \dots, \alpha_n$
> - Bilinear form $\langle\, ,\, \rangle : V \times V \longrightarrow \mathbb{R}$
>
> $$\langle \alpha_i, \alpha_j \rangle = -\cos\left(\frac{\pi}{m_{ij}}\right)$$

> **Geom. Repn.**
> $W \mapsto GL(V)$
> $s_i \mapsto \sigma_i$
> $\sigma_i(v) = v - 2\langle v, \alpha_i \rangle \alpha_i$
> "Reflection" by "hyperplane" "orthogonal" to α_i

Think: "angle" between hyperplanes i and j is π/m_{ij}, so
s_is_j = rot. by $2\pi/m_{ij}$
$(s_is_j)^{m_{ij}} = e$

$(m_{ij} = \infty \rightarrow \langle \alpha_i, \alpha_j \rangle = -1)$
$(m_{ii} = 1 \rightarrow \langle \alpha_i, \alpha_i \rangle = 1)$

$\boxed{\text{Prop The map } s_i \mapsto \sigma_i \text{ extends uniquely to a homomorphism } W \to GL(V).}$

Note I "proved" this incorrectly in lecture 8.

Uniqueness is clear, existence needs $(\sigma_i\sigma_j)^{m_{ij}} = e$.

Lemma. For $1 \le i < j \le n$ let

$$V_{ij} = \operatorname{span}(\alpha_i, \alpha_j)$$
$$V_{ij}^\perp = \{v \in V \mid \langle v, w\rangle = 0 \text{ for } w \in V_{ij}\}$$

Then $V = V_{ij} \oplus V_{ij}^\perp$. (For $m_{ij} \neq \infty$)

Pf. Say $i,j = 1,2$

Existence: Given $v \in V$ we need

$$v = \lambda_i\alpha_i + \lambda_j\alpha_j + v^\perp$$

$$\Rightarrow \begin{cases} \langle \alpha_i, v\rangle = \lambda_i + \lambda_j c \\ \langle \alpha_j, v\rangle = \lambda_i c + \lambda_j \end{cases} \qquad c = -\cos\left(\frac{\pi}{m_{ij}}\right).$$

$\hookrightarrow$ solve for λ_i, λ_j. $(c \neq 1)$

$\hookrightarrow$ for $v^\perp$

Uniqueness: λ_i, λ_j are determined as above ∎

By lemma, $\sigma_i\sigma_j$ acts "independently" on V_{ij} and $V_{ij}^\perp$.

On $V_{ij}^\perp$:

$$\sigma_i(v) = v - 2\langle v, \alpha_i\rangle\alpha_i = v$$
$$\sigma_j(v) = v$$
$$(\sigma_i\sigma_j)^{m_{ij}}v = v$$

On V_{ij}:

$$\sigma_i(\lambda_i\alpha_i + \lambda_j\alpha_j) = \lambda_i\alpha_i + \lambda_j\alpha_j - 2\langle\lambda_i\alpha_i + \lambda_j\alpha_j, \alpha_i\rangle\alpha_i$$
$$= (-\lambda_i - 2c\lambda_j)\alpha_i + \lambda_j\alpha_j$$

$$\sigma_i\begin{bmatrix}\lambda_i\\\lambda_j\end{bmatrix} = \begin{bmatrix}-1 & -2c\\0 & 1\end{bmatrix}\begin{bmatrix}\lambda_i\\\lambda_j\end{bmatrix} \qquad \sigma_j\begin{bmatrix}\lambda_i\\\lambda_j\end{bmatrix} = \begin{bmatrix}1 & 0\\-2c & -1\end{bmatrix}\begin{bmatrix}\lambda_i\\\lambda_j\end{bmatrix}$$

$$\sigma_i\sigma_j \text{ acts as } \begin{bmatrix}-1 & -2c\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\-2c & -1\end{bmatrix} = \begin{bmatrix}-1+4c^2 & 2c\\-2c & -1\end{bmatrix}$$

eigenvalues $e^{\frac{2\pi i}{m_{ij}}}$

So $(\sigma_i\sigma_j)^{m_{ij}} = e$ ∎

Note: This shows $\sigma_i\sigma_j$ has order m_{ij}.

(Similar argument when $m_{ij} = \infty$)

$\boxed{\text{Theorem The order of } s_is_j \text{ in } W \text{ is } m_{ij}.}$

Pf If s_is_j had smaller order in W, it would also in the geom repn. ∎

Later:

Will show $W \xrightarrow{\sigma} GL(V)$ is faithful (fiel) (injective)