So any two elts of W have a meet

Now let $A \subseteq W$ be arbitrary. $\bigwedge A = ?$

Take $x_1 \in A$ — Is $x_1 \leq_R$ all of A?

If so $x_1 = \bigwedge A$.

If not $x_1 \not\leq_R y_1$ for $y_1 \in A$

Let $x_2 = x_1 \wedge y_1$ $(x_2 < x_1)$ — Is $x_2 \leq_R$ all of A?

If so $x_2 = \bigwedge A$

If not $x_2 \not\leq_R y_2$ for $y_2 \in A$

Let $x_3 = x_2 \wedge y_2$ … $(x_3 < x_2)$

The descending chain $x_1 > x_2 > x_3 > \cdots$ must end since W has finite height, and the last x_k is $\bigwedge A$. ∎

Very similar to arguments for Noetherian rings

<u>Prop</u> If W is finite, the weak order is a lattice

<u>Pf</u> Any A has at least one upper bound w_0.

$$\bigvee A = \bigwedge (\text{upper bounds of } A)$$ ∎

This gives W another algebraic structure!

semilattice $\longleftrightarrow$ set with a binary operation which is
- commutative $\quad x \wedge y = y \wedge x$
- associative $\quad x \wedge (y \wedge z) = (x \wedge y) \wedge z$
- idempotent $\quad x \wedge x = x$

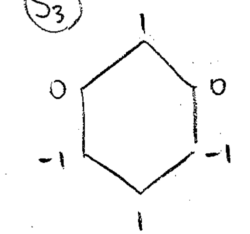
The semilattice structure and the group structure determine each other in most interesting cases.

(E.g., sufficient to require no ∞ ∞

<u>Thm.</u>

$$\mu_R(u, v) = \begin{cases} (-1)^J & \text{if } v = u w_0(J) \text{ for some } J \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

S_3



Topological proof → see book

Combinatorial proof?