Prop: $w \mapsto w w_0$, $w \mapsto w_0 w$ are order reversing; $w \mapsto w_0 w w_0$ is order preserving.

Pf. • We proved $T_L(ww_0) = T - T_L(w)$ ✓

• $w_0 u \trianglelefteq_R w_0 v$

$\Updownarrow$

$w_0 v = w_0 u s$ $\qquad \ell(w_0 v) < \ell(w_0 u s)$

$\Updownarrow$

$v = us$ $\qquad \ell(v) > \ell(us)$

$\Updownarrow$

$v \trianglerighteq_R u$ ✓

• Combine these

Prop Intervals of the weak order are finite.

The elts v of $[u,w]_R$ have

$T_L(u) \subset T_L(v) \subset T_L(w)$

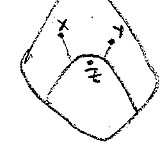
and $T_L(v) \neq T_L(v')$ for $v \neq v'$ (or else $v \leq_R v'$, $v' \leq_R v$) ∎

Theorem The weak order is a complete meet-semilattice.

complete: any nonempty (maybe infinite) set has a meet.

Pf First take $x, y \in W$, $E = [e,x] \cap [e,y]$. Induct on $\ell(x)$.

Take z of max length in E.



Claim $w \in E \Rightarrow w \leq_R z$

∘ First for generators

$w = s$: Sup $s \leq_R x$, $s \leq_R y$, $s \not\leq z$.

Take $z = s_1 \cdots s_k$
$x = s_1 \cdots s_k s_1' \cdots s_i'$ reduced
$y = s_1 \cdots s_k s_1'' \cdots s_j''$

$sx < x \rightarrow sx = s_1 \cdots s_k s_1' \cdots \widehat{s_a'} \cdots s_i'$

$x = \underbrace{s s_1 \cdots s_k} s_1' \cdots \widehat{s_a'} \cdots s_i'$

$y = \underbrace{s s_1 \cdots s_k} s_1'' \cdots \widehat{s_b''} \cdots s_j''$ reduced

$\rightarrow sz \leq_R x$, $sz \leq_R y$ $\qquad \ell(sz) > \ell(z)$ $\Rightarrow\Leftarrow$

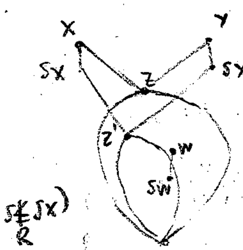
∘ Now for general w.

Take $w \leq_R x$, $w \leq_R y$. Take $s \leq_R w$.

Then $s \leq_R x$, $s \leq_R y \Rightarrow s \leq_R z$

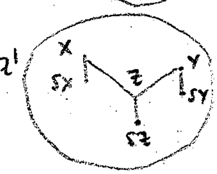
Then $sx \leq_R x$, $sy \leq_R y$, $sz \leq_R z$

Let $z' = sx \wedge sy$. ($s \not\leq_R z'$ since $s \not\leq_R sx$)



$\left\{ \begin{array}{l} z' \leq_R sx \\ z' \leq_R sy \end{array} \right. \xrightarrow{\text{lift}} \begin{array}{l} sz' \leq_R x \\ sz' \leq_R y \end{array} \rightarrow \ell(sz') \leq \ell(z)$

But also by "lifting", $sz \leq_R sx, sy \Rightarrow sz \leq_R z'$



$\Rightarrow sz = z' \Rightarrow$ lift $sw \leq z'$ to $w \leq_R z$.