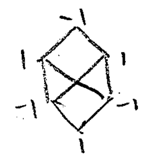
Möbius function

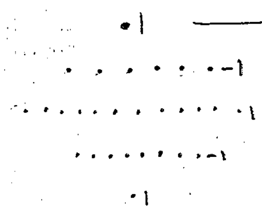
The Möbius function of a poset P is defined by

$$\mu(x,y) = \begin{cases} 1 & x = y \\ -\sum_{x \le z < y} \mu(x,z) & x < y \end{cases}$$

Ex: $\mu(\hat{0}, y)$:



Ex: face poset of



$= -(-F+E-V+1)$
$= 1$
by Euler's formula

$\mu(x,y)$ is the "Euler characteristic" of $[x,y]$.

It generalizes:

- Inclusion-Exclusion
- Möbius Inversion from number theory
- Euler's formula $V-E+F=2$
- ⋮

Prop Let P be graded. TFAE:

① $\mu(x,y) = (-1)^{r(y)-r(x)}$ for all $x \le y$.

② In $[x,y]$ $(x<y)$ the # of elts of even rank $=$ # of elts of odd rank

Such P are called "Eulerian".

Pf

② ⇒ ①: By induction on $r(y)-r(x)$.

$$0 = \sum_{x \le z \le y} \mu(x,z) = \underbrace{\sum_{x \le z \le y} (-1)^{r(z)-r(x)}}_{\substack{\text{equal number} \\ \text{of 1 and -1}}} + \mu(x,y) - (-1)^{r(y)-r(x)}$$

$$= \mu(x,y) - (-1)^{r(y)-r(x)}$$

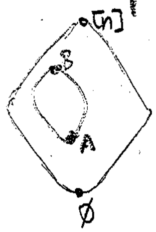
① ⇒ ②:

$$0 = \sum_{x \le z \le y} \mu(x,z) = \underbrace{\sum_{x \le z} (-1)^{r(z)-r(x)}}_{\text{so equal number of 1 and -1.}}$$ ∎

In HW3 you prove $\boxed{\text{Bruhat intervals are Eulerian.}}$

Examples of Eulerian posets

① Boolean poset of subsets of [n]



$$\sum_{A\subseteq C\subseteq B} (-1)^{|C|-|A|} = (1+-1)^{|B|-|A|} = 0$$

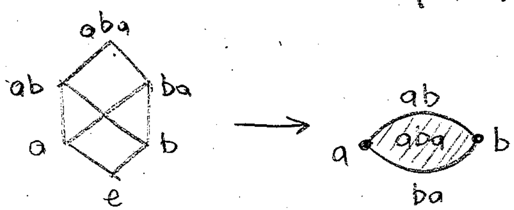
② Face poset of a convex polytope

(By Euler's formula.
Reason: "Euler characteristic" of a ball is 1)

③ Bruhat intervals

↓

Question: Are Bruhat intervals face posets of convex polytopes?



Conj. (Fomin-Shapiro) Bruhat intervals are face posets of "CW-complexes" of balls.

Hersh announced a proof (Dec 07)