The proposition gives us a "projection" map

$$P^J : W \to W^J$$

$$w \mapsto w^J \quad \text{such that} \quad w = w^J w_J \quad \begin{pmatrix} w^J \in W^J \\ w_J \in W_J \end{pmatrix}$$

$\boxed{\underline{\text{Prop}}\ \ P^J \text{ is order-preserving}}$

<u>Pf</u> Need: $w_1 \le w_2 \Rightarrow w_1^J \le w_2^J$

Induct on $\ell(w_2)$. $\ell(w_2) = \ell(w_1)$ is ok.

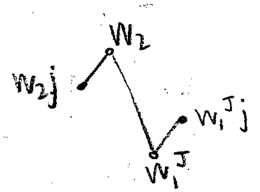
Sup true for lengths $< \ell(w_2)$.

$w_1^J \le w_1 \le w_2$ so done unless $w_2^J \ne w_2$

$\Rightarrow$ Assume $w_2^J \ne w_2 \Rightarrow w_2 = w_2^J (w_2)_J$

$$\ell(w_2) = \ell(w_2^J) + \ell((w_2)_J)$$

Let j = (some) last letter of $(w_2)_J$. $\Rightarrow w_2 j < w_2$



$\longrightarrow w_1^J \le w_2 j$

$\big\downarrow$ induction

$(w_1^J)^J \le (w_2 j)^J$

$w_1^J \le w_2^J$ ∎

$w_2 j = \underbrace{w_2^J}\underbrace{(w_2)_J j}$

<u>Q</u> What about $w \mapsto w_J$?

An enumerative application: length-generating fns

The Poincaré series of W is

$$W(q) = \sum_{w \in W} q^{\ell(w)}$$

Ex. $S_3(q) = 1 + 2q + 2q^2 + q^3$

$S_4(q) = 1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6$

$\boxed{\underline{\text{Prop.}}\ \ W(q) \text{ is symmetric for } W \text{ finite}}$ $\boxed{W(\tfrac{1}{q}) = \frac{W(q)}{q^{|T|}}}$

<u>Pf</u> $w \mapsto w w_0$ is an antiautomorphism.

$\boxed{\underline{\text{Lemma}}\ \ W(q) = W^J(q)\, W_J(q)}$

<u>Pf</u> $\text{RHS} = \sum_{w^J \in W^J} q^{\ell(w^J)} \sum_{w_J \in W_J} q^{\ell(w_J)}$

$$= \sum_{\substack{w^J \in W^J \\ w_J \in W_J}} q^{\ell(w^J w_J)} = \text{LHS} \quad ∎$$

So if I can compute $W^J(q)$ I can recursively compute $W(q)$.

Ex $W = S_n$

$J = \{s_1, \dots, s_{n-2}\}$

S_n^J: elts whose reduced expressions cannot end in $s_1, \dots, s_{n-2}$

$= \{e, s_{n-1}, s_{n-2}s_{n-1}, \dots, s_1 \cdots s_{n-1}\}$

$S_n^J(q) = 1 + q + q^2 + \dots + q^{n-1}$

$\to$ $S_n(q) = S_n^J(q)\,(S_n)_J(q)$

$= (1 + q + \dots + q^{n-1})\, S_{n-1}(q)$

$\to$ $\boxed{\text{Thm. } S_n(q) = (1+q)(1+q+q^2)\cdots(1+q+\dots+q^{n-1})}$

$= [n]! = [n][n-1]\cdots[2][1]$

where $[i]_q = 1 + q + \dots + q^{i-1}$

In fact,

$W = W_1 \times \dots \times W_n \Rightarrow W(q) = W_1(q) \cdots W_n(q)$ (easy)

and

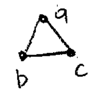
Thm If (W,S) is finite, irreducible, and $n = |S|$, there are "exponents" $e_1, \dots, e_n \in \mathbb{N}$ such that
$$W(q) = \prod_{i=1}^{n} [e_i + 1]_q$$
So $|W| = \prod_{i=1}^{n} (e_i + 1)$ and $|T| = \ell(w_0) = \sum_{i=1}^{n} e_i$

Pf: Conceptually $\to$ invariant theory

Manually $\to$ inductively, using the classification of finite Coxeter groups.

$$\boxed{\text{Thm} \quad \sum_{J \subseteq S} \frac{(-1)^{|J|}}{W_J(q)} = \begin{cases} \dfrac{q^{\ell(w_0)}}{W(q)} & W \text{ finite} \\ 0 & W \text{ infinite} \end{cases}}$$

Note: This allows recursive computation of $W(q)$.

Ex: $W =$ 

J	$W_J(q)$
$\emptyset$	1
a, b, c	$1+q$
ab, bc, ac	$(1+q+q^2)(1+q)$
abc	?

$1 - \dfrac{3}{1+q} + \dfrac{3}{(1+q)(1+q+q^2)} - \dfrac{1}{W(q)} = 0$

$W(q) = \dfrac{1 + 2q + 2q^2 + q^3}{1 - q - q^2 + q^3}$ $\leftarrow$ for infinite W, $W(1/q) = \pm W(q)$ also "symmetric"

Pf Mult by $W(q)$.

$\sum_{J \subseteq S} (-1)^{|J|} W^J(q) = \sum_{J \subseteq S} \sum_{w \in W^J} (-1)^{|J|} q^{\ell(w)}$

$= \sum_{w \in W} q^{\ell(w)} \sum_{\substack{J \subseteq S \\ w \in W^J}} (-1)^{|J|} = \sum_{w \in W} q^{\ell(w)} \sum_{J \subseteq S \setminus D(w)} (-1)^{|J|}$

$\uparrow$ descents of w = possible last letters of reduced words for w = s's that shorten w

$= \sum_{w \in W} q^{\ell(w)} (1 + (-1))^{S \setminus D(w)}$

$= \sum_{\substack{w \in W \\ D(w) = S}} q^{\ell(w)} = \begin{cases} q^{\ell(w_0)} & W \text{ finite} \\ 0 & W \text{ infinite} \end{cases}$